Valley Regime Study Notes

1 Model and graph construction (one-way shortcut)

- Nodes are labeled $1, \ldots, N$ (paper convention). Internally we use $0, \ldots, N-1$.
- $K = 2k$ neighbors on a ring: each node connects to $\pm 1, \ldots, \pm k$ with uniform jump probability $1/K$ (Eq. 1).
- Target node is $n = N/2$ (even N). Start node $n_0 = 1$.
- Single **directed** shortcut (paper indexing): $(n_0 + 5) \to (n + 1)$, i.e. $6 \to (N/2 + 1)$ (one-way, matching the paper).
- Internal graph data: padded neighbor table, degrees, directed edge list with weights $1/\deg(\mathrm{src})$.

2 Exact dynamics

- Analytical AW inversion of the generating function is the default solver (max steps fixed to 2000 for stability).
- The numerical master-equation iterator is kept only as a validation baseline.
- Randomness: MC runs use fixed seeds for reproducibility. Main MC uses seed=0; heatmap storage uses seed+123.

3 Peak and valley detection (Fig. 3 rule)

- Peak at t if $A(t-1) < A(t) > A(t+1)$ and $A(t) > 10^{-7}$.
- If a second peak exists, it must be at least 1% of the highest peak.
- For $K = 2$ we coarse-grain two steps to suppress parity micro-peaks.
- Valley time t_{valley} is the $\arg\min$ of $A(t)$ between the first two peaks.

4 Monte Carlo (joblib)

- Vectorized batch simulator; batches spawned with independent seeds. Uses `loky` when allowed; falls back to sequential in restricted environments.
- First-passage times are log-binned; density is smoothed with a 7-point moving average for clearer early-time behavior.

5 Classification of trajectories

- Let the first two peaks be at $t_1 < t_2$, with valley time t_{valley} between them, and define $\Delta = \max\{1, \lfloor 0.05(t_2 - t_1) \rfloor\}$.

- Classes by first-passage time T (valley + second-peak based split):

$$\begin{aligned}
\text{direct:} \quad & T < t_{\text{valley}} - \Delta, \\
\text{valley:} \quad & |T - t_{\text{valley}}| \le \Delta, \\
\text{intermediate:} \quad & t_{\text{valley}} + \Delta < T \le t_2 + \Delta, \\
\text{indirect:} \quad & T > t_2 + \Delta.
\end{aligned}$$

- Note: the usual "second-peak window" is $T \in [t_2 - \Delta, t_2 + \Delta]$, which is contained in the intermediate class.
- Heatmaps use full-path MC storage (up to 50k stored paths from a large MC run), split by class.

6 Scan setup

- Even $N \in [10, 160]$, $K \in \{2, 4, 6, 8\}$, $\rho = 1$.
- Bimodality ranges detected under the Fig. 3 rule are printed to stdout.

7 Scan summary (directed shortcut)

- Even $N \in [10, 160]$, $K \in \{2, 4, 6, 8\}$, $\rho = 1$.
- Bimodality detected: $K = 2$ none; $K = 4$: $N = 48..52$; $K = 6$: $N = 68..104$; $K = 8$: $N = 14, 88..160$.
- Explicitly: $K = 2$ still shows no two-peak structure under the Fig. 3 rule in this range.
- Additional check for $K = 2$: with directed shortcut $6 \to N/2 + 1$, coarse-graining by 2 steps (parity fix), and offsets $\{1, 2, 3, 4, 5\}$ for even $N \in [6, 80]$, no bimodality was found (second peak never reaches the 1% height criterion).
- Visualization: Figure 1 marks each (N, K) where two peaks exist under the Fig. 3 rule.

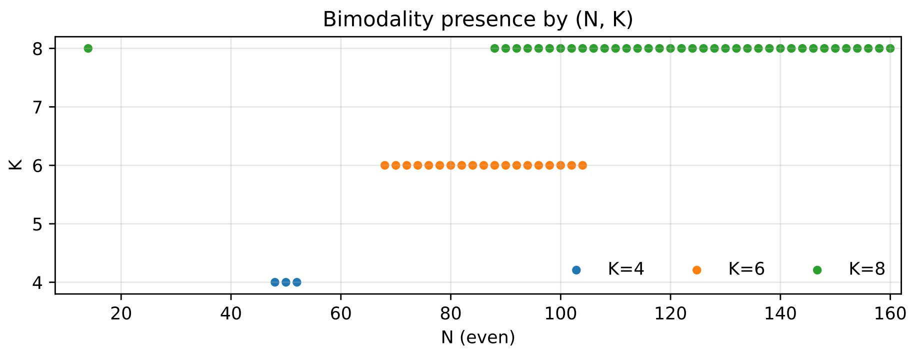

Figure 1: Bimodality map for $K \in \{2, 4, 6, 8\}$, even $N \in [10, 160]$.

8 Single case (N=100, K=6, $\rho = 1$)

8.1 First-passage distribution

8.2 Class definitions and proportions

- Peaks: $t_1 = 14$, $t_2 = 101$; valley at $t_{\text{valley}} = 74$; $\Delta = 4$.
- Classes by T: direct $< t_{\text{valley}} - \Delta$, valley $|T - t_{\text{valley}}| \le \Delta$, intermediate $t_{\text{valley}} + \Delta < T \le t_2 + \Delta$, indirect $> t_2 + \Delta$.
- MC proportions are shown in Figure 3.

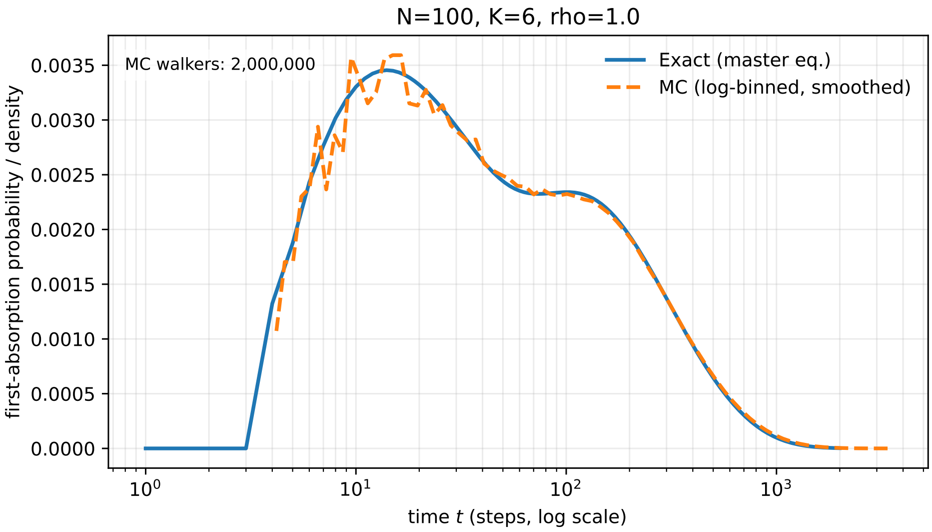

Figure 2: Exact vs. MC first-absorption distribution (MC: log-binned, 7-pt smoothed).

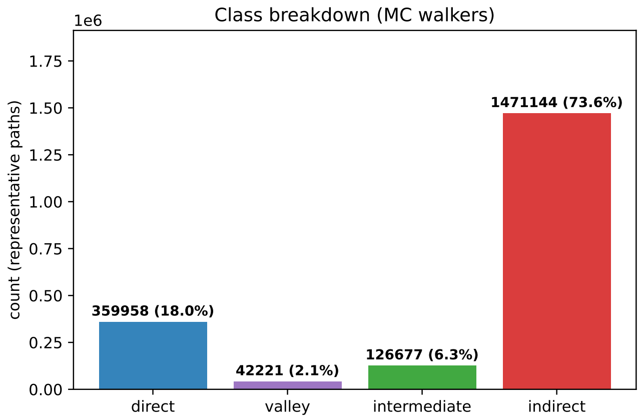

Figure 3: Class breakdown from the MC ensemble.

8.3 Shortcut usage (full MC)

Fractions of walkers using the shortcut edge by class: see Figure 4. Exact fractions are stored in `data/N100K6_metrics.json`.

8.4 Trajectory density heatmaps

- Construction: accumulate visits (time, node) for each class from stored full paths (up to 50k total stored paths from the main MC run, then split by class), per-time normalization to probabilities; Gaussian smoothing ($\sigma = 1$); clip at 99th percentile and PowerNorm to highlight weak signals; bilinear interpolation for visual continuity.
- Overlays: target line (cyan), shortcut endpoints (gold points at 5% of the time axis to indicate connectivity).

9 Data exports and seeds

- Scan raw data: `data/bimodality_scan.json` (details for each K, N, peaks, valley).

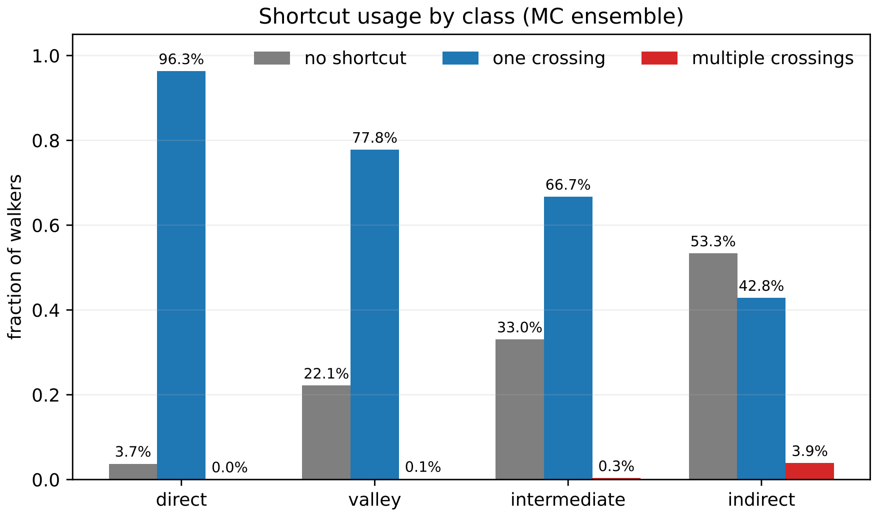

Figure 4: Shortcut usage by class (MC ensemble).

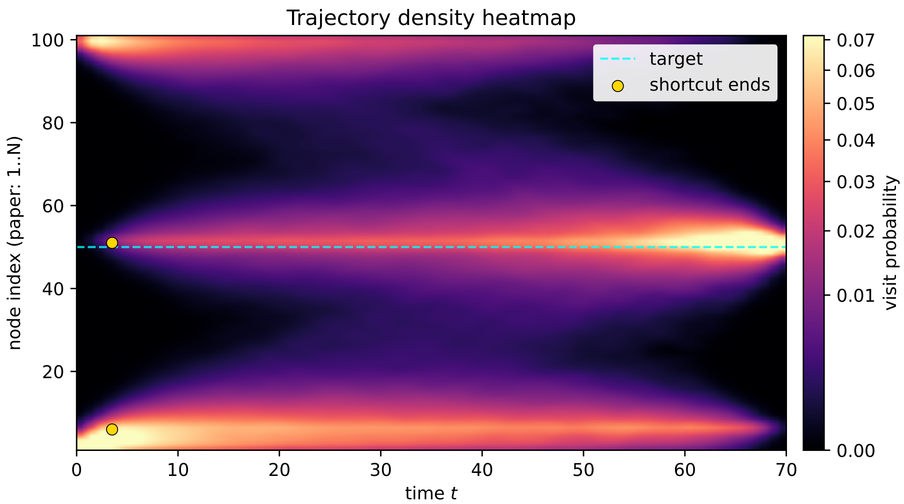

Figure 5: Heatmap: direct class visit density.

- Case metrics: `data/N100K6_metrics.json` (peaks, valley, Δ, MC class counts, shortcut usage); curves: `data/N100K6_curves.npz` (exact $A(t)$ and MC binned density).
- Seeds: main MC seed = 0; heatmap storage seed = seed+123; AW max steps = 2000.

10 Data appendix (key values)

- Scan summary (one-way shortcut $6 \to N/2+1$, $\rho = 1$): $K = 2$ none; $K = 4$: $N = 48..52$; $K = 6$: $N = 68..104$; $K = 8$: $N = 14, 88..160$. Detailed per-N records in `data/bimodality_scan.json`.
- N=100, K=6, $\rho = 1$: peaks $t_1 = 14$, $t_2 = 101$, valley $t_{\text{valley}} = 74$, $\Delta = 4$ (stored in `data/N100K6_metrics.json`).
- MC class proportions: see `figures/N100K6_class_counts_mc.pdf`; shortcut usage fractions stored in `data/N100K6_metrics.json`.
- Curves: `data/N100K6_curves.npz` contains $(t, A_{\text{exact}}(t))$ and MC log-binned density (centers, smoothed density) used in the main distribution plot.

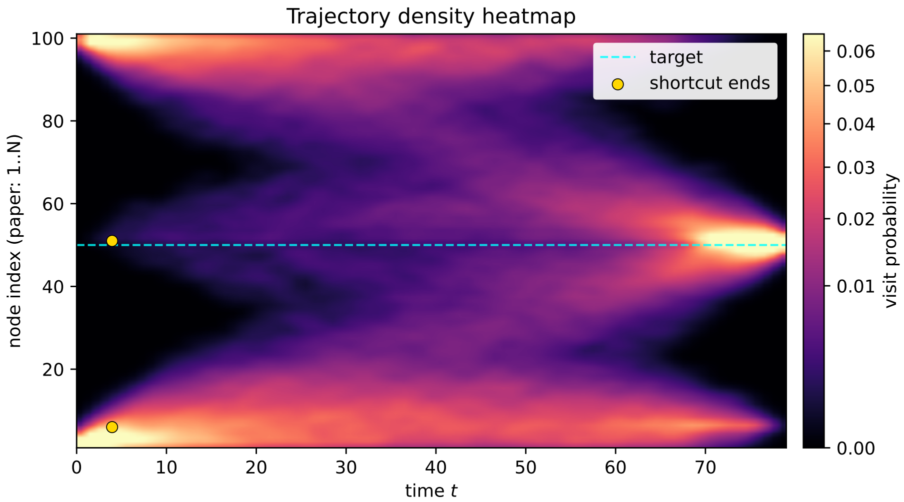

Figure 6: Heatmap: valley class visit density.

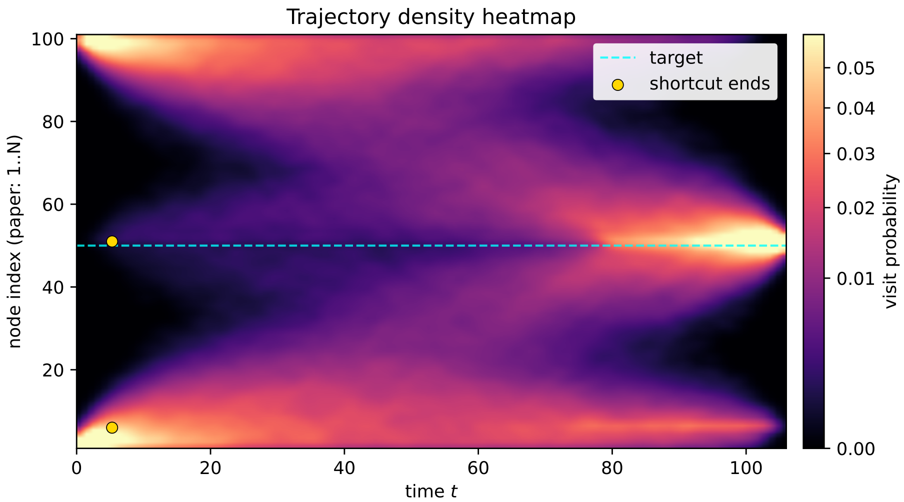

Figure 7: Heatmap: intermediate class visit density.

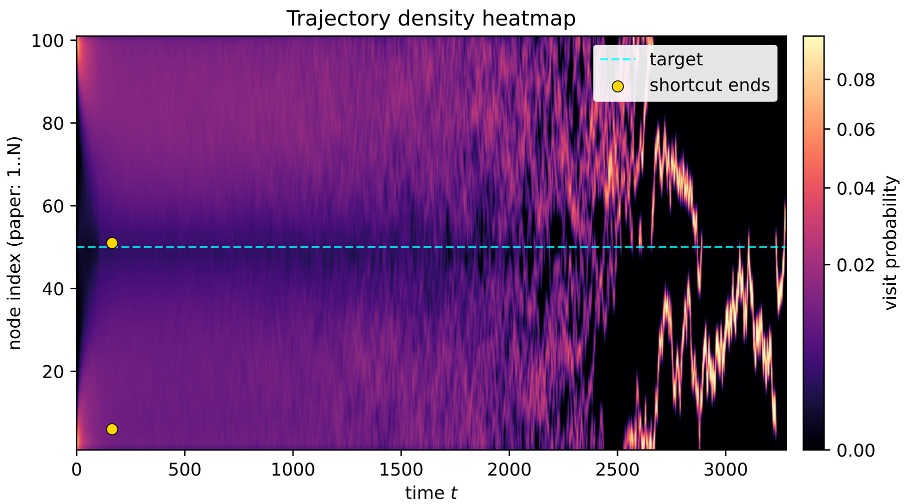

Figure 8: Heatmap: indirect class visit density.