Exact propagators and first-passage generating functions on a 1D ring
with one directed long-range link:

(A) lazy-reservoir shortcut vs (B) nearest-neighbour rewiring shortcut
Giuggioli–Bristol notation: $\underline{B}$ (ring), $\underline{A}$ (defect), $\underline{X} = \underline{B} - \underline{A}$, defect parameters η

Overview

We derive, in full detail:

- the defect-free generating function (Green's function) $\tilde{Q}_{n_0}(n, z)$ for a *lazy* nearest-neighbour random walk on a finite ring of size N;
- the exact defective generating function $\tilde{S}_{n_0}(n, z)$ after introducing a *single directed* long-range link using two constructions:
  (A) **Lazy-reservoir shortcut**: the shortcut probability is taken from the self-loop at the source node;
  (B) **Nearest-neighbour rewiring**: the shortcut probability is taken from an existing nearest-neighbour jump at the source node.
- two complementary derivations for $\tilde{S}$: (i) a low-rank (Sherman–Morrison) resolvent update; (ii) the Giuggioli defect-technique determinant formula (scalar for (A), explicit 2×2 for (B));
- the first-passage generating function $\tilde{F}_{n_0 \to n}(z) = \tilde{S}_{n_0}(n, z)/\tilde{S}_n(n, z)$, including a fully explicit Chebyshev closed form for construction (A);
- a detailed comparison of the two constructions and their relation to Watts–Strogatz small-world rewiring.
- numerical validation against direct matrix inversion (double precision).

Throughout we use the Giuggioli–Bristol defect-technique notation ($\underline{B}, \underline{A}, \underline{X}, \eta$), and we keep *all* intermediate steps, including the explicit $k = 0$ cancellation in the Chebyshev identity step.

1 Notation and model

1.1 Ring, indexing, and transition matrices

We consider a discrete-time random walk on a 1D ring (cycle) of size N with periodic boundary conditions (PBC). Sites are labeled $n \in \{0, 1, \dots, N-1\}$ and all indices are understood modulo N.

We use the column-stochastic convention

$$\Pr\big(X_{t+1} = n \mid X_t = n'\big) = B_{n,n'} \quad \text{(defect-free ring)},$$
$$\Pr\big(X_{t+1} = n \mid X_t = n'\big) = A_{n,n'} \quad \text{(with defect)}.$$

Thus $P(t+1) = \underline{B}\, P(t)$ for the defect-free process, and $\sum_n B_{n,n'} = \sum_n A_{n,n'} = 1$ for each column n'.

The defect matrix is

$$\underline{X} \equiv \underline{B} - \underline{A}.$$

In the Supplement notation, each defect *pair* (u_i, v_i) carries two parameters η_{u_i,v_i} and η_{v_i,u_i}, which correspond to the off-diagonal entries of $\underline{X}$:

$$\eta_{u,v} \equiv X_{v,u} = B_{v,u} - A_{v,u}, \qquad \eta_{v,u} \equiv X_{u,v} = B_{u,v} - A_{u,v}.$$

1.2 Baseline (defect-free) lazy nearest-neighbour ring

Let $q \in (0,1)$ be the *total* nearest-neighbour jump probability (nearest-neighbour range $k=1$). Then the defect-free transition matrix $\underline{B}$ is

$$B_{n,n} = 1 - q, \qquad B_{n\pm1,n} = \frac{q}{2}, \qquad B_{m,n} = 0 \text{ otherwise.}$$

In words: at each step, with probability q the walker moves to one of the two neighbours (left/right with probability $q/2$ each), and with probability $1-q$ the walker stays in place (self-loop / sojourn).

1.3 Propagators and generating functions (Green's functions)

- Defect-free propagator: $Q_{n_0}(n,t) = \Pr(X_t = n \mid X_0 = n_0)$ under $\underline{B}$.
- Defective propagator: $S_{n_0}(n,t) = \Pr(X_t = n \mid X_0 = n_0)$ under $\underline{A}$.
- Generating function (discrete-time z-transform): $\tilde{f}(z) = \sum_{t=0}^{\infty} f(t)\, z^t$.

We write $\tilde{Q}_{n_0}(n,z) = \sum_{t\geq 0} Q_{n_0}(n,t)\, z^t$ and $\tilde{S}_{n_0}(n,z)$ similarly.

Matrix-wise,

$$\underline{G}_B(z) \equiv (\underline{I} - z\underline{B})^{-1} = \sum_{t=0}^{\infty} z^t \underline{B}^t, \qquad \underline{G}_A(z) \equiv (\underline{I} - z\underline{A})^{-1} = \sum_{t=0}^{\infty} z^t \underline{A}^t,$$

and therefore

$$\tilde{Q}_{n_0}(n,z) = [\underline{G}_B(z)]_{n,n_0}, \qquad \tilde{S}_{n_0}(n,z) = [\underline{G}_A(z)]_{n,n_0}.$$

1.4 Ring distance and Chebyshev polynomials

For any two sites a, b define the minimal ring distance

$$d(a,b) \equiv \min\bigl(|a-b|,\ N - |a-b|\bigr) \in \left\{0, 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor\right\}.$$

Chebyshev polynomials of the first and second kind are

$$\mathcal{T}_n(x) = \cos(n \arccos x), \qquad \mathcal{U}_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}.$$

2 Part I — Defect-free propagator $\tilde{Q}_{n_0}(n,z)$ in Chebyshev form

2.1 Step 1: z-transform of the master equation (fully explicit)

The defect-free master equation is

$$Q_{n_0}(n,t+1) = \frac{q}{2}\,Q_{n_0}(n-1,t) + \frac{q}{2}\,Q_{n_0}(n+1,t) + (1-q)\,Q_{n_0}(n,t), \tag{1}$$

with initial condition $Q_{n_0}(n,0) = \delta_{n,n_0}$ and PBC.

Multiply both sides of (1) by z^t and sum $t = 0$ to ∞:

$$\sum_{t=0}^{\infty} Q_{n_0}(n,t+1)z^t = \frac{q}{2}\sum_{t=0}^{\infty} Q_{n_0}(n-1,t)z^t + \frac{q}{2}\sum_{t=0}^{\infty} Q_{n_0}(n+1,t)z^t + (1-q)\sum_{t=0}^{\infty} Q_{n_0}(n,t)z^t.$$

The sums on the right are the generating functions:

$$\sum_{t=0}^{\infty} Q_{n_0}(n\pm1,t)z^t = \tilde{Q}_{n_0}(n\pm1,z), \qquad \sum_{t=0}^{\infty} Q_{n_0}(n,t)z^t = \tilde{Q}_{n_0}(n,z).$$

For the left-hand side we use the time-shift identity:

$$\begin{aligned}
\sum_{t=0}^{\infty} Q_{n_0}(n,t+1)z^t &= \frac{1}{z}\sum_{t=0}^{\infty} Q_{n_0}(n,t+1)z^{t+1} = \frac{1}{z}\sum_{s=1}^{\infty} Q_{n_0}(n,s)z^s \\
&= \frac{1}{z}\left(\sum_{s=0}^{\infty} Q_{n_0}(n,s)z^s - Q_{n_0}(n,0)\right) = \frac{1}{z}\left(\tilde{Q}_{n_0}(n,z) - \delta_{n,n_0}\right).
\end{aligned}$$

Substituting into the summed master equation yields

$$\frac{1}{z}\left(\tilde{Q}_{n_0}(n,z) - \delta_{n,n_0}\right) = \frac{q}{2}\,\tilde{Q}_{n_0}(n-1,z) + \frac{q}{2}\,\tilde{Q}_{n_0}(n+1,z) + (1-q)\,\tilde{Q}_{n_0}(n,z).$$

Multiplying by z and rearranging gives the Helmholtz-type equation

$$\big[1 - z(1-q)\big]\,\tilde{Q}_{n_0}(n,z) - \frac{qz}{2}\left(\tilde{Q}_{n_0}(n-1,z) + \tilde{Q}_{n_0}(n+1,z)\right) = \delta_{n,n_0}. \tag{2}$$

2.2 Step 2: discrete Fourier representation on the ring

Let

$$\theta_k \equiv \frac{2\pi k}{N}, \qquad k = 0,1,\ldots,N-1.$$

The eigenvalues of the ring transition matrix $\underline{B}$ are

$$\lambda_k = 1 - q + q\cos\theta_k.$$

Therefore the generating function (Green's function) is

$$\tilde{Q}_{n_0}(n,z) = \frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i\theta_k(n-n_0)}}{1 - z\lambda_k} = \frac{1}{N}\sum_{k=0}^{N-1} \frac{e^{i\theta_k(n-n_0)}}{1 - z\left[1 - q + q\cos\theta_k\right]}. \tag{3}$$

By translational invariance, $\tilde{Q}_{n_0}(n,z)$ depends only on $d = d(n,n_0)$.

2.3 Step 3: rewrite the sum in the form required by Identity E3

Introduce the spectral parameter

$$\sigma(z) \equiv \frac{1 - z(1-q)}{qz}. \tag{4}$$

Then

$$1 - z[1 - q + q\cos\theta_k] = qz\big(\sigma(z) - \cos\theta_k\big).$$

Insert this into (3) and use the ring symmetry (pair k with $N-k$) to write the numerator as a cosine:

$$\tilde{Q}_{n_0}(n,z) = \frac{1}{qz}\,\frac{1}{N}\sum_{k=0}^{N-1}\frac{\cos\!\big(d\,\theta_k\big)}{\sigma(z) - \cos\theta_k}, \qquad d = d(n,n_0). \tag{5}$$

2.4 Step 4: apply Identity E3 and show the $k=0$ cancellation explicitly

Identity E3 (periodic case, Appendix E in the 2020 reference) states that for $m \in \{0,1,\ldots,N\}$,

$$\sum_{k=1}^{N-1}\frac{\cos\big(\frac{2\pi mk}{N}\big)}{\sigma - \cos\big(\frac{2\pi k}{N}\big)} = \frac{1}{1-\sigma} + N\,\frac{\mathcal{T}_{N-m}(\sigma) + \mathcal{T}_m(\sigma)}{(\sigma^2-1)\,\mathcal{U}_{N-1}(\sigma)}. \tag{6}$$

Align the cosine. Using $\theta_k = \frac{2\pi k}{N}$, we write

$$\cos(d\theta_k) = \cos\left(d\frac{2\pi k}{N}\right) = \cos\left(\frac{2\pi(m=d)\,k}{N}\right).$$

Thus we apply (6) with

$$m = d, \qquad \sigma = \sigma(z),$$

which yields for the partial sum $k = 1,\ldots,N-1$:

$$\sum_{k=1}^{N-1}\frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \frac{1}{1-\sigma(z)} + N\,\frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2-1)\,\mathcal{U}_{N-1}(\sigma(z))}. \tag{7}$$

Restore the $k=0$ term. We now write the full sum from (5) as

$$\sum_{k=0}^{N-1}\frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \underbrace{\frac{\cos(0)}{\sigma(z) - \cos(0)}}_{k=0\text{ term}} + \sum_{k=1}^{N-1}\frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k}.$$

Since $\cos(0) = 1$, the missing $k=0$ term equals $1/(\sigma(z)-1)$. Using (7),

$$\sum_{k=0}^{N-1}\frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \frac{1}{\sigma(z)-1} + \frac{1}{1-\sigma(z)} + N\,\frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2-1)\,\mathcal{U}_{N-1}(\sigma(z))}. \tag{8}$$

The first two terms cancel explicitly:

$$\frac{1}{\sigma(z)-1} + \frac{1}{1-\sigma(z)} = \frac{1}{\sigma(z)-1} - \frac{1}{\sigma(z)-1} = 0.$$

Therefore the full sum becomes

$$\sum_{k=0}^{N-1}\frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = N\,\frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2-1)\,\mathcal{U}_{N-1}(\sigma(z))}. \tag{9}$$

Substitute (9) back into (5) to obtain the closed form:

$$\boxed{\tilde{Q}_{n_0}(n,z) = \frac{\mathcal{T}_d(\sigma(z)) + \mathcal{T}_{N-d}(\sigma(z))}{qz\,\big(\sigma(z)^2-1\big)\,\mathcal{U}_{N-1}(\sigma(z))}, \qquad d = d(n,n_0).} \tag{10}$$

2.5 Odd vs even N (explicit)

Let $d = d(n, n_0) \in \{0, 1, \dots, \lfloor N/2 \rfloor\}$.

- If N is odd, (10) holds as written for all d.
- If N is even and $d = N/2$ (antipodes), then $N - d = d$ and the numerator becomes $2\mathcal{T}_{N/2}(\sigma(z))$:

$$\tilde{Q}_{n_0}(n, z)\big|_{d=N/2} = \frac{2\,\mathcal{T}_{N/2}(\sigma(z))}{qz\left(\sigma(z)^2 - 1\right)\mathcal{U}_{N-1}(\sigma(z))}.$$

3 Part II — One directed shortcut: two constructions and their exact $\tilde{S}$

3.1 A unifying algebraic view: "probability-mass transfer" in one column

Both constructions we consider are *local* modifications of a single column of $\underline{B}$: we pick a *source* node u, and move probability mass p from some *old destination* b to a *new destination* a. At the level of transition matrices this is

$$\boxed{\underline{A} = \underline{B} + p\,(|a\rangle - |b\rangle)\,\langle u|\,.} \tag{11}$$

Here $|a\rangle$ denotes the basis vector at site a.

Validity constraints. The only constraint is that the modified transition probabilities remain nonnegative. If b is the self-loop ($b = u$), then $p \le B_{u,u} = 1 - q$. If b is a nearest neighbour ($b = u \pm 1$), then $p \le B_{u\pm1,u} = q/2$.

3.2 Sherman–Morrison derivation of the resolvent update (all steps)

Define the defect-free and defective resolvents

$$\underline{G}_B(z) = (\underline{I} - z\underline{B})^{-1}, \qquad \underline{G}_A(z) = (\underline{I} - z\underline{A})^{-1}.$$

Using (11), we write

$$\underline{I} - z\underline{A} = \underline{I} - z\underline{B} - zp(|a\rangle - |b\rangle)\,\langle u| \equiv \underline{M} - \boldsymbol{u}\boldsymbol{v}^\top,$$

with

$$\underline{M} \equiv \underline{I} - z\underline{B}, \qquad \boldsymbol{u} \equiv zp(|a\rangle - |b\rangle), \qquad \boldsymbol{v}^\top \equiv \langle u|\,.$$

We now invert $\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top$ exactly.

Claim (Sherman–Morrison for a rank-1 update). If $\underline{M}$ is invertible and $1 - \boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u} \neq 0$, then

$$(\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top)^{-1} = \underline{M}^{-1} + \frac{\underline{M}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1}}{1 - \boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u}}. \tag{12}$$

Verification. Let $\underline{R}$ denote the right-hand side of (12). Compute $(\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top)\underline{R}$:

$$\begin{aligned}(\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top)\underline{R} &= (\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top)\underline{M}^{-1} + (\underline{M} - \boldsymbol{u}\boldsymbol{v}^\top)\frac{\underline{M}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1}}{1 - \boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u}} \\ &= \underline{I} - \boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1} + \frac{\boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1}}{1 - \boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u}} - \frac{\boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u}\boldsymbol{v}^\top\underline{M}^{-1}}{1 - \boldsymbol{v}^\top\underline{M}^{-1}\boldsymbol{u}}.\end{aligned}$$

Factor $\boldsymbol{u}\boldsymbol{v}^\top \underline{M}^{-1}$ in the last two terms:

$$\frac{\boldsymbol{u}\boldsymbol{v}^\top \underline{M}^{-1}}{1-\boldsymbol{v}^\top \underline{M}^{-1}\boldsymbol{u}}\left[1-(\boldsymbol{v}^\top \underline{M}^{-1}\boldsymbol{u})\right] = \boldsymbol{u}\boldsymbol{v}^\top \underline{M}^{-1}.$$

Thus the last two terms together equal $+\boldsymbol{u}\boldsymbol{v}^\top \underline{M}^{-1}$, which cancels the $-\boldsymbol{u}\boldsymbol{v}^\top \underline{M}^{-1}$ term. Therefore $(\underline{M}-\boldsymbol{u}\boldsymbol{v}^\top)\underline{R} = \underline{I}$, proving (12).

Apply to the random walk. Since $\underline{M}^{-1} = (\underline{I}-z\underline{B})^{-1} = \underline{G}_B(z)$, (12) gives

$$\underline{G}_A(z) = \underline{G}_B(z) + \frac{zp\,\underline{G}_B(z)\,(|a\rangle - |b\rangle)\,\langle u|\,\underline{G}_B(z)}{1-zp\,\langle u|\,\underline{G}_B(z)\,(|a\rangle - |b\rangle)}. \tag{13}$$

Take the matrix element (n, n_0) to obtain the defective propagator generating function:

$$\boxed{\tilde{S}_{n_0}(n,z) = \tilde{Q}_{n_0}(n,z) + \frac{zp\,[\tilde{Q}_a(n,z)-\tilde{Q}_b(n,z)]\,\tilde{Q}_{n_0}(u,z)}{1-zp\,[\tilde{Q}_a(u,z)-\tilde{Q}_b(u,z)]}.} \tag{14}$$

This formula is *exact* and will be specialised to the two constructions below.

3.3 Construction (A): shortcut probability taken from the self-loop (lazy reservoir)

Definition of the defect. We add a directed shortcut $u \to v$ of probability p by taking p from the self-loop at u:

$$A_{v,u} = B_{v,u} + p = p, \qquad A_{u,u} = B_{u,u} - p = (1-q) - p, \qquad A_{u\pm1,u} = B_{u\pm1,u} = \frac{q}{2}.$$

This is (11) with

$$a = v, \qquad b = u, \qquad 0 \le p \le 1-q.$$

Exact propagator (rank-1 form). Specialising (14) to $(a = v, b = u)$ gives

$$\boxed{\tilde{S}^{(\mathrm{A})}_{n_0}(n,z) = \tilde{Q}_{n_0}(n,z) + \frac{zp\,[\tilde{Q}_v(n,z)-\tilde{Q}_u(n,z)]\,\tilde{Q}_{n_0}(u,z)}{1-zp\,[\tilde{Q}_v(u,z)-\tilde{Q}_u(u,z)]}.} \tag{15}$$

Using translational invariance on the ring, $\tilde{Q}_x(y,z) = \tilde{Q}(d(x,y),z)$, this can be written entirely in terms of distances:

$$\boxed{\tilde{S}^{(\mathrm{A})}_{n_0}(n,z) = \tilde{Q}(d(n,n_0),z) - \frac{zp\,[\tilde{Q}(d(n,u),z)-\tilde{Q}(d(n,v),z)]\,\tilde{Q}(d(u,n_0),z)}{1+zp\,[\tilde{Q}(0,z)-\tilde{Q}(d(u,v),z)]}.} \tag{16}$$

Defect-technique encoding (one defect pair). In terms of $\underline{X} = \underline{B} - \underline{A}$, the only nonzero off-diagonal entry is

$$X_{v,u} = -p,$$

so the Supplement defect parameter is

$$\eta_{u,v} = X_{v,u} = -p, \qquad \eta_{v,u} = 0,$$

and the diagonal change $X_{u,u} = +p$ is implied by column-stochasticity.

Recovering (A) from the defect-technique determinant formula ($M = 1$). Although (A) is already solved by the rank-1 update (15)–(16), it is useful to connect to the Giuggioli defect-technique formula (especially since (B) will naturally appear as "multiple defects"). For M defect pairs (u_i, v_i), the defect-technique result is

$$\tilde{S}_{n_0}(n,z) = \tilde{Q}_{n_0}(n,z) - 1 + \frac{\det\big[\underline{H}(n,n_0;z)\big]}{\det\big[\underline{H}(z)\big]}. \tag{17}$$

For a single defect pair (u, v) we have $M = 1$ and the determinants reduce to scalars. With the standard definitions

$$\det[\underline{H}(z)] = H(z) = \eta_{u,v}\,\tilde{Q}_{(u-v)}(u,z) - \eta_{v,u}\,\tilde{Q}_{(u-v)}(v,z) - \frac{1}{z}, \tag{18}$$

$$\det[\underline{H}(n,n_0;z)] = H(n,n_0;z) = H(z) - \tilde{Q}_{(u-v)}(n,z)\Big[\eta_{u,v}\,\tilde{Q}_{n_0}(u,z) - \eta_{v,u}\,\tilde{Q}_{n_0}(v,z)\Big], \tag{19}$$

where $\tilde{Q}_{(u-v)}(x,z) \equiv \tilde{Q}_u(x,z) - \tilde{Q}_v(x,z)$, and inserting $\eta_{u,v} = -p$, $\eta_{v,u} = 0$ gives

$$\det[\underline{H}(z)] = -\frac{1}{z} - p\Big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\Big), \tag{20}$$

$$\det[\underline{H}(n,n_0;z)] = \det[\underline{H}(z)] + p\Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,\tilde{Q}(d(u,n_0),z). \tag{21}$$

Substituting (20)–(21) into (17) reproduces exactly (16).

3.4 Construction (B): rewiring probability from a nearest-neighbour jump to the shortcut

Definition of the defect. Choose one nearest neighbour of u as the *old destination* w; for definiteness on the ring we can take $w = u+1$ (the "clockwise" neighbour). We create a directed shortcut $u \to v$ of probability p by *reducing* the probability of the jump $u \to w$ by p:

$$\begin{aligned} A_{v,u} &= B_{v,u} + p = p, \\ A_{w,u} &= B_{w,u} - p = \frac{q}{2} - p, \\ A_{u,u} &= B_{u,u} = 1 - q, \\ A_{u-1,u} &= B_{u-1,u} = \frac{q}{2}. \end{aligned}$$

This is (11) with

$$a = v, \qquad b = w = u + 1, \qquad 0 \le p \le \frac{q}{2}.$$

(The case $p = q/2$ corresponds to a *full* rewiring of the entire clockwise edge probability onto the shortcut.)

Exact propagator (rank-1 form). Specialising (14) to $(a = v, b = w)$ gives

$$\boxed{\tilde{S}^{(\mathrm{B})}_{n_0}(n,z) = \tilde{Q}_{n_0}(n,z) + \frac{zp\,[\tilde{Q}_v(n,z) - \tilde{Q}_w(n,z)]\,\tilde{Q}_{n_0}(u,z)}{1 - zp\,[\tilde{Q}_v(u,z) - \tilde{Q}_w(u,z)]}.} \tag{22}$$

In distance form (using $d(u,w) = 1$ on the ring):

$$\boxed{\tilde{S}^{(\mathrm{B})}_{n_0}(n,z) = \tilde{Q}(d(n,n_0),z) + \frac{zp\,[\tilde{Q}(d(n,v),z) - \tilde{Q}(d(n,w),z)]\,\tilde{Q}(d(u,n_0),z)}{1 - zp\,[\tilde{Q}(d(u,v),z) - \tilde{Q}(1,z)]}.} \tag{23}$$

3.5 How construction (B) appears as "multiple defects" in the η-pair formalism (explicit 2×2 determinant)

Although (22) is rank-1 at the matrix level, in the Supplement pair notation it is convenient to represent (B) as the superposition of *two* defect pairs, because the self-loop at u is *unchanged* and must emerge as a cancellation.

Two defect pairs. Write (B) as "add p to $u \to v$ and subtract p from the self-loop" (pair 1) *plus* "subtract p from $u \to w$ and add p to the self-loop" (pair 2). This corresponds to two pairs:

$$(u_1, v_1) = (u, v), \qquad (u_2, v_2) = (u, w),$$

with defect parameters

$$\eta_{u_1,v_1} = \eta_{u,v} = -p, \qquad \eta_{u_2,v_2} = \eta_{u,w} = +p, \qquad \eta_{v_1,u_1} = \eta_{v,u} = 0, \qquad \eta_{v_2,u_2} = \eta_{w,u} = 0.$$

Indeed, this yields off-diagonal changes

$$X_{v,u} = \eta_{u,v} = -p, \qquad X_{w,u} = \eta_{u,w} = +p,$$

and diagonal change $X_{u,u} = -(\eta_{u,v} + \eta_{u,w}) = 0$ (so $A_{u,u} = B_{u,u}$), as required.

Build $\underline{H}(z)$ explicitly. For $M = 2$ defect pairs, the Supplement defines $\underline{H}(z)$ by

$$\begin{aligned} H(z)_{i,j} &= \eta_{u_i,v_i}\, \tilde{Q}_{(u_j - v_j)}(u_i, z) - \eta_{v_i,u_i}\, \tilde{Q}_{(u_j - v_j)}(v_i, z) - \frac{\delta_{i,j}}{z}, \\ \tilde{Q}_{(u_j - v_j)}(x, z) &= \tilde{Q}_{u_j}(x, z) - \tilde{Q}_{v_j}(x, z). \end{aligned}$$

Here $\eta_{v_i,u_i} = 0$, and $u_1 = u_2 = u$, so

$$\tilde{Q}_{(u-v)}(u, z) = \tilde{Q}_u(u, z) - \tilde{Q}_v(u, z) = \tilde{Q}(0, z) - \tilde{Q}(d(u, v), z),$$

$$\tilde{Q}_{(u-w)}(u, z) = \tilde{Q}_u(u, z) - \tilde{Q}_w(u, z) = \tilde{Q}(0, z) - \tilde{Q}(d(u, w), z) = \tilde{Q}(0, z) - \tilde{Q}(1, z).$$

Therefore

$$\underline{H}(z) = \begin{pmatrix} -\frac{1}{z} - p\big(\tilde{Q}(0, z) - \tilde{Q}(d(u, v), z)\big) & -p\big(\tilde{Q}(0, z) - \tilde{Q}(1, z)\big) \\ +p\big(\tilde{Q}(0, z) - \tilde{Q}(d(u, v), z)\big) & -\frac{1}{z} + p\big(\tilde{Q}(0, z) - \tilde{Q}(1, z)\big) \end{pmatrix}.$$

Its determinant is computed step-by-step:

$$\begin{aligned} \det[\underline{H}(z)] &= \frac{1}{z^2} + \frac{p}{z}\Big[(\tilde{Q}(0) - \tilde{Q}_{uv}) - (\tilde{Q}(0) - \tilde{Q}_1)\Big] \\ &= \frac{1}{z^2} + \frac{p}{z}\big(\tilde{Q}_1 - \tilde{Q}_{uv}\big). \end{aligned}$$

where $\tilde{Q}_{uv} \equiv \tilde{Q}(d(u, v), z)$ and $\tilde{Q}_1 \equiv \tilde{Q}(1, z)$. The p^2 terms cancel exactly. Thus

$$\boxed{\det[\underline{H}(z)] = \frac{1}{z^2}\Big[1 + zp\big(\tilde{Q}(1, z) - \tilde{Q}(d(u, v), z)\big)\Big] = \frac{1}{z^2}\Big[1 - zp\big(\tilde{Q}_{uv} - \tilde{Q}_1\big)\Big].} \tag{24}$$

Build $\underline{H}(n, n_0; z)$ and its determinant. The Supplement defines

$$H(n, n_0; z)_{i,j} = H(z)_{i,j} - \tilde{Q}_{(u_j - v_j)}(n, z)\Big[\eta_{u_i,v_i}\tilde{Q}_{n_0}(u_i, z) - \eta_{v_i,u_i}\tilde{Q}_{n_0}(v_i, z)\Big].$$

Since $\eta_{v_i,u_i} = 0$ and $u_i = u$ for both i, the bracket reduces to $\eta_{u_i,v_i}\tilde{Q}_{n_0}(u, z)$. Define

$$\begin{aligned}
Q_{un_0}(z) &\equiv \tilde{Q}_{n_0}(u, z),\\
\Delta_v(n) &\equiv \tilde{Q}_{(u-v)}(n, z) = \tilde{Q}_u(n, z) - \tilde{Q}_v(n, z),\\
\Delta_w(n) &\equiv \tilde{Q}_{(u-w)}(n, z) = \tilde{Q}_u(n, z) - \tilde{Q}_w(n, z).
\end{aligned}$$

Then

$$\underline{H}(n, n_0; z) = \begin{pmatrix} H_{11} + p\,\Delta_v(n)\,Q_{un_0}(z) & H_{12} + p\,\Delta_w(n)\,Q_{un_0}(z) \\ H_{21} - p\,\Delta_v(n)\,Q_{un_0}(z) & H_{22} - p\,\Delta_w(n)\,Q_{un_0}(z) \end{pmatrix},$$

and a direct expansion gives (keeping every term)

$$\det[\underline{H}(n, n_0; z)] = \det[\underline{H}(z)] + \frac{p}{z}\,Q_{un_0}(z)\,\big(\Delta_w(n) - \Delta_v(n)\big).$$

Finally note that $\Delta_w(n) - \Delta_v(n) = \tilde{Q}_v(n, z) - \tilde{Q}_w(n, z)$, so

$$\boxed{\det[\underline{H}(n, n_0; z)] = \frac{1}{z^2}\Big[1 - zp\big(\tilde{Q}_{uv} - \tilde{Q}_1\big)\Big] + \frac{p}{z}\,\tilde{Q}_{n_0}(u, z)\,\big(\tilde{Q}_v(n, z) - \tilde{Q}_w(n, z)\big).} \tag{25}$$

Recover $\tilde{S}^{(\mathrm{B})}$ from the determinant formula. The defect technique gives

$$\tilde{S}_{n_0}(n, z) = \tilde{Q}_{n_0}(n, z) - 1 + \frac{\det[\underline{H}(n, n_0; z)]}{\det[\underline{H}(z)]}.$$

Substituting (24)–(25) yields

$$-1 + \frac{\det[\underline{H}(n, n_0; z)]}{\det[\underline{H}(z)]} = \frac{zp\,\tilde{Q}_{n_0}(u, z)\,(\tilde{Q}_v(n, z) - \tilde{Q}_w(n, z))}{1 - zp(\tilde{Q}_{uv} - \tilde{Q}_1)},$$

which reproduces exactly (22).

4 Chebyshev packaging for the defective propagators

Define

$$\sigma = \sigma(z), \qquad \mathcal{C}(z) \equiv qz\big(\sigma^2 - 1\big)\mathcal{U}_{N-1}(\sigma), \qquad \mathcal{N}_d(\sigma) \equiv \mathcal{T}_d(\sigma) + \mathcal{T}_{N-d}(\sigma),$$

so that (10) is

$$\tilde{Q}(d, z) = \frac{\mathcal{N}_d(\sigma)}{\mathcal{C}(z)}.$$

Then every term in (16) and (23) is explicit in $\mathcal{T}_n, \mathcal{U}_{N-1}$.

For example, in construction (B) the denominator becomes

$$1 - zp\big(\tilde{Q}(d(u, v), z) - \tilde{Q}(1, z)\big) = 1 - \frac{zp}{\mathcal{C}(z)}\Big(\mathcal{N}_{d(u,v)}(\sigma) - \mathcal{N}_1(\sigma)\Big),$$

and similarly for the numerator differences.

5 Part III — First-passage generating function

For either construction, the first-passage generating function to site n starting from n_0 is

$$\boxed{\tilde{F}_{n_0\to n}(z) = \frac{\tilde{S}_{n_0}(n,z)}{\tilde{S}_n(n,z)}.} \tag{26}$$

5.1 Construction (A): explicit form

Insert (16) for the numerator and the same formula with $n_0 = n$ for the denominator:

$$\tilde{F}^{(\mathrm{A})}_{n_0\to n}(z) = \frac{\tilde{S}^{(\mathrm{A})}_{n_0}(n,z)}{\tilde{S}^{(\mathrm{A})}_n(n,z)}.$$

All occurrences of $\tilde{Q}$ are given explicitly by (10).

Chebyshev closed form (construction (A)). Starting from (26) with (16), define the two recurring blocks

$$D_Q(z) \equiv 1 + zp\big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\big), \qquad \Delta_n(z) \equiv \tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z). \tag{27}$$

Multiplying numerator and denominator by $D_Q(z)$ yields the equivalent form without nested fractions:

$$\boxed{\tilde{F}^{(\mathrm{A})}_{n_0\to n}(z) = \frac{\tilde{Q}(d(n,n_0),z)\, D_Q(z) - zp\,\Delta_n(z)\,\tilde{Q}(d(u,n_0),z)}{\tilde{Q}(0,z)\, D_Q(z) - zp\,\Delta_n(z)\,\tilde{Q}(d(u,n),z)}.} \tag{28}$$

Now substitute the Chebyshev packaging from (10):

$$\tilde{Q}(d,z) = \frac{\mathcal{N}_d(\sigma(z))}{\mathcal{C}(z)}, \qquad \mathcal{C}(z) \equiv qz\big(\sigma(z)^2 - 1\big)\mathcal{U}_{N-1}(\sigma(z)), \qquad \mathcal{N}_d(\sigma) \equiv \mathcal{T}_d(\sigma) + \mathcal{T}_{N-d}(\sigma),$$

and introduce the compact factor

$$\boxed{D(z) \equiv \mathcal{C}(z) + zp\big(\mathcal{N}_0(\sigma(z)) - \mathcal{N}_{d(u,v)}(\sigma(z))\big),} \tag{29}$$

so that $D_Q(z) = D(z)/\mathcal{C}(z)$. Cancelling the common $\mathcal{C}(z)^{-2}$ factor gives the fully explicit closed form

$$\boxed{\tilde{F}^{(\mathrm{A})}_{n_0\to n}(z) = \frac{\mathcal{N}_{d(n,n_0)}(\sigma(z))\, D(z) - zp\left(\mathcal{N}_{d(n,u)}(\sigma(z)) - \mathcal{N}_{d(n,v)}(\sigma(z))\right)\mathcal{N}_{d(u,n_0)}(\sigma(z))}{\mathcal{N}_0(\sigma(z))\, D(z) - zp\left(\mathcal{N}_{d(n,u)}(\sigma(z)) - \mathcal{N}_{d(n,v)}(\sigma(z))\right)\mathcal{N}_{d(u,n)}(\sigma(z))}.} \tag{30}$$

Here $\mathcal{N}_0(\sigma) = \mathcal{T}_0(\sigma) + \mathcal{T}_N(\sigma) = 1 + \mathcal{T}_N(\sigma)$.

5.2 Construction (B): explicit form

Insert (23) for the numerator and the same formula with $n_0 = n$ for the denominator:

$$\tilde{F}^{(\mathrm{B})}_{n_0\to n}(z) = \frac{\tilde{S}^{(\mathrm{B})}_{n_0}(n,z)}{\tilde{S}^{(\mathrm{B})}_n(n,z)}.$$

Again, all terms reduce to Chebyshev polynomials via (10).

6 Part IV — Detailed comparison and relation to Watts–Strogatz SWN rewiring

6.1 Mechanistic comparison: where does the shortcut probability come from?

(A) Lazy-reservoir shortcut (self-loop transfer).

- Probability mass is taken from the self-loop at the source: $u \to u$ decreases, $u \to v$ increases.
- Constraint: $0 \le p \le 1 - q$.
- Nearest-neighbour probabilities $u \to u \pm 1$ remain exactly $q/2$, so the local diffusive step statistics are unchanged.
- In the η-pair formalism this is one defect pair: $M = 1$.

(B) Nearest-neighbour rewiring shortcut (edge-to-edge transfer).

- Probability mass is taken from an existing jump $u \to w$ (with $w = u + 1$ or $u - 1$) and moved to $u \to v$.
- Constraint: $0 \le p \le q/2$ for partial rewiring of a single neighbour edge; the full rewiring limit is $p = q/2$.
- The self-loop probability at u remains $1-q$, but the local jump kernel at u becomes biased if only one neighbour is rewired.
- In the η-pair formalism it is naturally represented by *two* defect pairs (u, v) and (u, w) with opposite signs, so $M = 2$, even though the net matrix update is still rank-1.

6.2 Relation to Watts–Strogatz SWN

In a classical Watts–Strogatz small-world network (SWN), one rewires edges of an underlying ring lattice. If the random walk chooses uniformly among neighbours, then the transition probabilities are proportional to $1/\chi_n$ (node degree). A single rewiring event changes the degree of (at least) the target node and the removed node, hence *all* outgoing transition probabilities from those nodes must be renormalised, producing *multiple* nonzero entries in $X = B - A$ and thus multiple defects in the determinant formalism. In contrast, constructions (A) and (B) above operate at the level of *transition probabilities* directly: they move a prescribed probability mass p within one column, so they remain low-rank and analytically tractable.

6.3 Consistency checks

- $p = 0$ in either construction gives $\underline{A} = \underline{B}$ and $\tilde{S} = \tilde{Q}$.
- In (A), the maximal shortcut is $p = 1 - q$ (all sojourn probability at u is converted into shortcut jumps).
- In (B), the full rewiring of the clockwise edge corresponds to $p = q/2$ (the entire $u \to w$ probability is redirected to $u \to v$).

Numerical validation (double precision)

We validated (10), (16), and (30) against direct matrix computation of the resolvent for construction (A) (lazy-reservoir shortcut). For each parameter set (N, q, p, u, v, z) we built the column-stochastic transition matrices $\underline{B}$ (ring) and $\underline{A}$ (directed shortcut), and computed

$$G_B(z) = (\underline{I} - z\underline{B})^{-1}, \qquad G_A(z) = (\underline{I} - z\underline{A})^{-1}.$$

The ground-truth generating functions are $\tilde{Q}_{n_0}(n, z) = [G_B(z)]_{n,n_0}$ and $\tilde{S}_{n_0}(n, z) = [G_A(z)]_{n,n_0}$. For first passage we used $\tilde{F}_{n_0 \to n}(z) = [G_A(z)]_{n,n_0}/[G_A(z)]_{n,n}$ for $n_0 \neq n$. We report the maximum absolute error over all indices:

$$\max_{n,n_0} |\Delta\tilde{Q}|, \qquad \max_{n,n_0} |\Delta\tilde{S}|, \qquad \max_{n_0 \neq n} |\Delta\tilde{F}|.$$

N	q	p	u	v	z	$\max\lvert\Delta\tilde{Q}\rvert$	$\max\lvert\Delta\tilde{S}\rvert$	$\max\lvert\Delta\tilde{F}\rvert$
9	0.6	0.2	1	4	0.3	2.22e-16	4.44e-16	5.55e-17
10	0.7	0.1	2	7	0.5	4.44e-16	2.22e-16	5.55e-17
12	0.5	0.3	0	6	0.6	4.44e-16	4.44e-16	8.33e-17
15	0.8	0.05	3	4	0.4	2.22e-16	4.44e-16	1.11e-16
20	0.4	0.2	5	13	0.75	1.33e-15	1.33e-15	1.67e-16

In a sweep of 200 random cases ($N \leq 40$, seed $= 0$, $q \in [0.2, 0.9]$, $z \in [0.2, 0.85]$, $p \in [0, 1-q]$), we found

$$\max|\Delta\tilde{Q}| = 3.11 \times 10^{-15}, \qquad \max|\Delta\tilde{S}| = 3.11 \times 10^{-15}, \qquad \max|\Delta\tilde{F}| = 4.44 \times 10^{-16}.$$

These residuals are consistent with floating-point round-off and the conditioning of matrix inversion as $z \to 1^-$.