Exact Generating Functions and First-Passage Statistics on a 1D Ring with One *Directed* Shortcut (defect technique; nearest-neighbour $k = 1$; $K = 2$ defect sites)

Notation: $(\underline{B}, \underline{A}, \underline{X}, \eta)$

Notation and model

Ring lattice. We consider a discrete-time random walk on a 1D ring (cycle) of size N with periodic boundary conditions (PBC). Sites are labeled $n \in \{0, 1, \dots, N-1\}$ and all indices are understood modulo N.

Transition matrices and convention. We use the convention

$$\Pr(X_{t+1} = n \mid X_t = n') = B_{n,n'} \qquad \text{(defect-free ring)},$$
$$\Pr(X_{t+1} = n \mid X_t = n') = A_{n,n'} \qquad \text{(with defect)}.$$

Thus, the master equation is $P(n, t+1) = \sum_{n'} B_{n,n'} P(n', t)$ for the defect-free process. Both $\underline{B}$ and $\underline{A}$ are column-stochastic: $\sum_n B_{n,n'} = \sum_n A_{n,n'} = 1$.

Baseline (defect-free) lazy random walk. Let $q \in (0, 1)$ be the *total* nearest-neighbour ($k = 1$) jump probability. Then

$$B_{n,n} = 1 - q, \qquad B_{n\pm1,n} = \frac{q}{2}, \qquad B_{m,n} = 0 \text{ otherwise}.$$

Remark on the notation k vs K. This document considers the nearest-neighbour ring (jumps only to $n \pm 1$, i.e. $k = 1$). The symbol $K = 2$ is used here *only* to count the two defect sites at the shortcut endpoints (u, v); it should not be confused with the connectivity parameter $K = 2k$ used in other ring-lattice/small-world models.

Single directed shortcut. We add *one directed* (unidirectional) shortcut from a site u to a site v. Its probability is denoted p and is *taken from the lazy weight at u*. Concretely, only the outgoing probabilities from u are changed:

$$A_{u,u} = 1 - q - p, \qquad A_{u\pm1,u} = \frac{q}{2}, \qquad A_{v,u} = B_{v,u} + p, \qquad A_{n,u} = B_{n,u} \text{ otherwise},$$

and for all $n' \neq u$ we keep $A_{n,n'} = B_{n,n'}$. The constraint is

$$0 \le p \le 1 - q$$

so that $A_{u,u} \ge 0$.

Defect matrix and defect parameters. We encode the modifications by the (sparse) defect matrix

$$\underline{X} \equiv \underline{B} - \underline{A}.$$

In the defect-technique formulas of the Supplement, the defect parameters associated with a pair (u_i, v_i) are written η_{u_i,v_i} and η_{v_i,u_i} and correspond to the off-diagonal entries of $\underline{X}$:

$$\eta_{u,v} \equiv X_{v,u} = B_{v,u} - A_{v,u}, \qquad \eta_{v,u} \equiv X_{u,v} = B_{u,v} - A_{u,v}.$$

For a *single directed* shortcut $u \to v$ of probability p (built from the lazy weight at u) we have

$$\boxed{\eta_{u,v} = -p, \qquad \eta_{v,u} = 0,}$$

while the corresponding diagonal change at u is implied by probability conservation: $X_{u,u} = B_{u,u} - A_{u,u} = p = -\eta_{u,v}$.

Propagators and generating functions.

- Defect-free (ring) propagator: $Q_{n_0}(n,t) = \Pr(X_t = n \mid X_0 = n_0)$ under $\underline{B}$.
- Defective (network) propagator: $S_{n_0}(n,t) = \Pr(X_t = n \mid X_0 = n_0)$ under $\underline{A}$.
- z-transform: $\tilde{f}(z) = \sum_{t=0}^{\infty} f(t)\, z^t$.

We denote $\tilde{Q}_{n_0}(n,z) = \sum_{t\geq 0} Q_{n_0}(n,t) z^t$ and similarly $\tilde{S}_{n_0}(n,z)$.

Ring distance. For any two sites a, b define

$$d(a,b) \equiv \min\big(|a-b|,\ N - |a-b|\big) \in \Big\{0, 1, \dots, \Big\lfloor \frac{N}{2} \Big\rfloor\Big\}.$$

Chebyshev polynomials. We use

$$\mathcal{T}_n(x) = \cos\big(n \arccos x\big), \qquad \mathcal{U}_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sin(\arccos x)}.$$

1 Part 1 — Homogeneous (defect-free) propagator $\tilde{Q}_{n_0}(n,z)$ on the ring

1.1 Generating function on a finite ring

The defect-free master equation is

$$Q_{n_0}(n,t+1) = \frac{q}{2}\, Q_{n_0}(n-1,t) + \frac{q}{2}\, Q_{n_0}(n+1,t) + (1-q)\, Q_{n_0}(n,t), \tag{1}$$

with $Q_{n_0}(n,0) = \delta_{n,n_0}$ and PBC.

Taking the z-transform of (1) proceeds as follows. Multiply both sides of (1) by z^t and sum from $t = 0$ to ∞:

$$\sum_{t=0}^{\infty} Q_{n_0}(n,t+1) z^t = \frac{q}{2} \sum_{t=0}^{\infty} Q_{n_0}(n-1,t) z^t + \frac{q}{2} \sum_{t=0}^{\infty} Q_{n_0}(n+1,t) z^t + (1-q) \sum_{t=0}^{\infty} Q_{n_0}(n,t) z^t.$$

The three sums on the right are just the generating functions:

$$\sum_{t=0}^{\infty} Q_{n_0}(n \pm 1, t) z^t = \tilde{Q}_{n_0}(n \pm 1, z), \qquad \sum_{t=0}^{\infty} Q_{n_0}(n,t) z^t = \tilde{Q}_{n_0}(n,z).$$

For the time-shifted left-hand side we use the identity

$$\begin{aligned}\sum_{t=0}^{\infty} Q_{n_0}(n,t+1)z^t &= \frac{1}{z}\sum_{t=0}^{\infty} Q_{n_0}(n,t+1)z^{t+1} = \frac{1}{z}\sum_{s=1}^{\infty} Q_{n_0}(n,s)z^s \\ &= \frac{1}{z}\left(\sum_{s=0}^{\infty} Q_{n_0}(n,s)z^s - Q_{n_0}(n,0)\right) = \frac{1}{z}\left(\tilde{Q}_{n_0}(n,z) - \delta_{n,n_0}\right),\end{aligned}$$

where we used $Q_{n_0}(n,0) = \delta_{n,n_0}$. Substituting these into the summed master equation yields

$$\begin{aligned}\frac{1}{z}\left(\tilde{Q}_{n_0}(n,z) - \delta_{n,n_0}\right) &= \frac{q}{2}\,\tilde{Q}_{n_0}(n-1,z) + \frac{q}{2}\,\tilde{Q}_{n_0}(n+1,z) \\ &\quad + (1-q)\,\tilde{Q}_{n_0}(n,z), \\ \tilde{Q}_{n_0}(n,z) - \delta_{n,n_0} &= \frac{qz}{2}\Big(\tilde{Q}_{n_0}(n-1,z) + \tilde{Q}_{n_0}(n+1,z)\Big) + z(1-q)\,\tilde{Q}_{n_0}(n,z), \\ \big[1 - z(1-q)\big]\tilde{Q}_{n_0}(n,z) &- \frac{qz}{2}\Big(\tilde{Q}_{n_0}(n-1,z) + \tilde{Q}_{n_0}(n+1,z)\Big) \\ &= \delta_{n,n_0}.\end{aligned}$$

Thus we obtain the Helmholtz-type equation

$$\big[1 - z(1-q)\big]\tilde{Q}_{n_0}(n,z) - \frac{qz}{2}\Big(\tilde{Q}_{n_0}(n-1,z) + \tilde{Q}_{n_0}(n+1,z)\Big) = \delta_{n,n_0}. \tag{2}$$

Diagonalize on the ring by discrete Fourier modes. Let

$$\theta_k \equiv \frac{2\pi k}{N}, \qquad k = 0, 1, \dots, N-1.$$

The eigenvalues of $\underline{B}$ are

$$\lambda_k = 1 - q + q\cos\theta_k.$$

Hence the Green's function (generating function of the propagator) is

$$\tilde{Q}_{n_0}(n,z) = \frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{i\theta_k(n-n_0)}}{1 - z\lambda_k} = \frac{1}{N}\sum_{k=0}^{N-1}\frac{e^{i\theta_k(n-n_0)}}{1 - z\left[1 - q + q\cos\theta_k\right]}. \tag{3}$$

By translational invariance, $\tilde{Q}_{n_0}(n,z)$ depends only on $d = d(n,n_0)$.

1.2 Preparing the sum for the Chebyshev identity

Introduce the spectral parameter

$$\sigma(z) \equiv \frac{1 - z(1-q)}{qz}. \tag{4}$$

Then

$$1 - z[1 - q + q\cos\theta_k] = qz\big(\sigma(z) - \cos\theta_k\big).$$

Insert this into (3) and use symmetry of the ring to write the numerator as a cosine:

$$\tilde{Q}_{n_0}(n,z) = \frac{1}{qz}\,\frac{1}{N}\sum_{k=0}^{N-1}\frac{\cos\big(d\,\theta_k\big)}{\sigma(z) - \cos\theta_k}, \qquad d = d(n,n_0). \tag{5}$$

1.3 Applying Identity E3 (Appendix E, 2020 paper) with the $k = 0$ term shown explicitly

Identity E3 for the periodic case reads (for $m \in \{0, 1, \ldots, N\}$)

$$\sum_{k=1}^{N-1} \frac{\cos\left(\frac{2\pi mk}{N}\right)}{\sigma - \cos\left(\frac{2\pi k}{N}\right)} = \frac{1}{1-\sigma} + N \, \frac{\mathcal{T}_{N-m}(\sigma) + \mathcal{T}_m(\sigma)}{(\sigma^2 - 1)\, \mathcal{U}_{N-1}(\sigma)}. \tag{6}$$

We now apply (6) to the sum in (5). First, we use $\theta_k = \frac{2\pi k}{N}$ to *align* the cosine:

$$\cos\!\big(d\,\theta_k\big) = \cos\!\left(d \frac{2\pi k}{N}\right) = \cos\!\left(\frac{2\pi (m = d)\, k}{N}\right).$$

Thus we set

$$m = d, \qquad \sigma = \sigma(z)$$

in (6), which gives an expression for the *partial* sum $k = 1, \ldots, N-1$:

$$\sum_{k=1}^{N-1} \frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \frac{1}{1 - \sigma(z)} + N \, \frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2 - 1)\, \mathcal{U}_{N-1}(\sigma(z))}. \tag{7}$$

Next, we restore the missing $k = 0$ term in the full sum of (5):

$$\sum_{k=0}^{N-1} \frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \underbrace{\frac{\cos(0)}{\sigma(z) - \cos(0)}}_{k=0 \text{ term}} + \sum_{k=1}^{N-1} \frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k}.$$

Since $\cos(0) = 1$, the $k = 0$ term is

$$\frac{\cos(0)}{\sigma(z) - \cos(0)} = \frac{1}{\sigma(z) - 1}.$$

Using (7), we obtain

$$\sum_{k=0}^{N-1} \frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = \frac{1}{\sigma(z) - 1} + \frac{1}{1 - \sigma(z)} + N \, \frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2 - 1)\, \mathcal{U}_{N-1}(\sigma(z))}. \tag{8}$$

The first two terms cancel *exactly*:

$$\frac{1}{\sigma(z) - 1} + \frac{1}{1 - \sigma(z)} = \frac{1}{\sigma(z) - 1} - \frac{1}{\sigma(z) - 1} = 0.$$

Therefore the full sum becomes

$$\sum_{k=0}^{N-1} \frac{\cos(d\theta_k)}{\sigma(z) - \cos\theta_k} = N \, \frac{\mathcal{T}_{N-d}(\sigma(z)) + \mathcal{T}_d(\sigma(z))}{(\sigma(z)^2 - 1)\, \mathcal{U}_{N-1}(\sigma(z))}. \tag{9}$$

Finally, substitute (9) back into (5) to obtain

$$\boxed{\tilde{Q}_{n_0}(n, z) = \frac{\mathcal{T}_d(\sigma(z)) + \mathcal{T}_{N-d}(\sigma(z))}{qz\,\big(\sigma(z)^2 - 1\big)\, \mathcal{U}_{N-1}(\sigma(z))}, \qquad d = d(n, n_0).} \tag{10}$$

1.4 Odd vs even N (explicit)

Let $d = d(n, n_0) \in \{0, 1, \ldots, \lfloor N/2 \rfloor\}$.

N **odd:** $N = 2M + 1$**.** Then $d \in \{0, 1, \ldots, M\}$ and $N - d \neq d$ unless $d = 0$. Equation (10) holds as written.

N **even:** $N = 2M$**.** Then $d \in \{0, 1, \ldots, M\}$. For $d < M$, (10) holds as written. For the antipodal case $d = M = N/2$, the two Chebyshev terms coincide. Indeed, the numerator in (10) is $\mathcal{T}_d(\sigma(z)) + \mathcal{T}_{N-d}(\sigma(z))$, and for even N with $d = N/2$ we have $N - d = d$, hence

$$N - d = d \quad \Rightarrow \quad \mathcal{T}_d(\sigma(z)) + \mathcal{T}_{N-d}(\sigma(z)) = \mathcal{T}_{N/2}(\sigma(z)) + \mathcal{T}_{N/2}(\sigma(z)) = 2\,\mathcal{T}_{N/2}(\sigma(z)).$$

Therefore,

$$\tilde{Q}_{n_0}(n,z)\big|_{d=N/2} = \frac{2\,\mathcal{T}_{N/2}(\sigma(z))}{qz\left(\sigma(z)^2 - 1\right)\mathcal{U}_{N-1}(\sigma(z))}.$$

2 Part 2 — Heterogeneous propagator $\tilde{S}_{n_0}(n,z)$ (one directed shortcut)

2.1 Defect-technique formula (matrix form)

For M (inert, probability-preserving) defect pairs (u_i, v_i) with defect parameters η_{u_i,v_i} and η_{v_i,u_i}, the generating function of the heterogeneous propagator is

$$\tilde{S}_{n_0}(n,z) = \tilde{Q}_{n_0}(n,z) - 1 + \frac{\det\!\big[\underline{H}(n,n_0;z)\big]}{\det\!\big[\underline{H}(z)\big]}. \tag{11}$$

The $M \times M$ matrices $\underline{H}(z)$ and $\underline{H}(n,n_0;z)$ are defined componentwise by

$$H(z)_{i,j} = \eta_{u_i,v_i}\,\tilde{Q}_{(u_j-v_j)}(u_i,z) - \eta_{v_i,u_i}\,\tilde{Q}_{(u_j-v_j)}(v_i,z) - \frac{\delta_{i,j}}{z}, \tag{12}$$

$$H(n,n_0;z)_{i,j} = H(z)_{i,j} - \tilde{Q}_{(u_j-v_j)}(n,z)\Big[\eta_{u_i,v_i}\,\tilde{Q}_{n_0}(u_i,z) - \eta_{v_i,u_i}\,\tilde{Q}_{n_0}(v_i,z)\Big], \tag{13}$$

with the difference notation

$$\tilde{Q}_{(u_j-v_j)}(x,z) \equiv \tilde{Q}_{u_j}(x,z) - \tilde{Q}_{v_j}(x,z).$$

2.2 Single directed shortcut (nearest-neighbour $k = 1$; $K = 2$ defect sites; $M = 1$ defect pair)

A single directed shortcut involves exactly two special sites, u and v (hence $K = 2$ defect sites; this K counts only the endpoints), but it corresponds to one defect *pair* in the formalism: $M = 1$ with $(u_1, v_1) = (u, v)$.

For a directed shortcut $u \to v$ of probability p taken from the lazy weight at u, the only nonzero off-diagonal defect parameter is

$$\eta_{u,v} = X_{v,u} = B_{v,u} - A_{v,u} = -p, \qquad \eta_{v,u} = 0.$$

Since $M = 1$, determinants reduce to scalars: $\det[\underline{H}] = H$.

2.3 Explicit scalar determinants

Using (12)–(13) with $M = 1$ gives

$$\det[\underline{H}(z)] = H(z) = \eta_{u,v}\,\tilde{Q}_{(u-v)}(u,z) - \eta_{v,u}\,\tilde{Q}_{(u-v)}(v,z) - \frac{1}{z}, \tag{14}$$

$$\det[\underline{H}(n,n_0;z)] = H(n,n_0;z) = H(z) - \tilde{Q}_{(u-v)}(n,z)\Big[\eta_{u,v}\,\tilde{Q}_{n_0}(u,z) - \eta_{v,u}\,\tilde{Q}_{n_0}(v,z)\Big]. \tag{15}$$

On the homogeneous ring, $\tilde{Q}_a(b,z)$ depends only on $d(a,b)$. In particular, letting $d_{uv} = d(u,v)$:

$$\tilde{Q}_{(u-v)}(u,z) = \tilde{Q}_u(u,z) - \tilde{Q}_v(u,z) = \tilde{Q}(0,z) - \tilde{Q}(d_{uv},z),$$

and

$$\tilde{Q}_{(u-v)}(n,z) = \tilde{Q}_u(n,z) - \tilde{Q}_v(n,z) = \tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z).$$

Also $\tilde{Q}_{n_0}(u,z) = \tilde{Q}(d(u,n_0),z)$.

With $\eta_{u,v} = -p$ and $\eta_{v,u} = 0$, (14)–(15) become

$$\boxed{\det[\underline{H}(z)] = -\frac{1}{z} - p\Big(\tilde{Q}(0,z) - \tilde{Q}(d_{uv},z)\Big),} \tag{16}$$

$$\boxed{\det[\underline{H}(n,n_0;z)] = \det[\underline{H}(z)] + p\Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,\tilde{Q}(d(u,n_0),z).} \tag{17}$$

2.4 Closed form for $\tilde{S}_{n_0}(n,z)$

Insert (16)–(17) into (11):

$$\begin{aligned}
\tilde{S}_{n_0}(n,z) &= \tilde{Q}_{n_0}(n,z) - 1 + \frac{\det[\underline{H}(n,n_0;z)]}{\det[\underline{H}(z)]} \\
&= \tilde{Q}_{n_0}(n,z) + \frac{p\Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,\tilde{Q}(d(u,n_0),z)}{\det[\underline{H}(z)]}.
\end{aligned}$$

Since $\det[\underline{H}(z)] = -(1/z)\big[1 + zp(\tilde{Q}(0,z) - \tilde{Q}(d_{uv},z))\big]$, we obtain the explicit rational form

$$\boxed{\begin{aligned}
\tilde{S}_{n_0}(n,z) = \tilde{Q}(d(n,n_0),z)& \\
- \frac{z\,p}{1 + z\,p\Big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\Big)}& \\
\times \Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,&\tilde{Q}(d(u,n_0),z).
\end{aligned}} \tag{18}$$

2.5 Determinants in Chebyshev form

Define, for compactness,

$$\sigma = \sigma(z), \qquad \mathcal{C}(z) \equiv qz\big(\sigma^2 - 1\big)\mathcal{U}_{N-1}(\sigma), \qquad \mathcal{N}_d(\sigma) \equiv \mathcal{T}_d(\sigma) + \mathcal{T}_{N-d}(\sigma).$$

Then (10) reads

$$\tilde{Q}(d,z) = \frac{\mathcal{N}_d(\sigma)}{\mathcal{C}(z)}.$$

Let $d_{ab} = d(a,b)$. Using (16)–(17) we obtain

$$\boxed{\begin{aligned}
\det[\underline{H}(z)] &= -\frac{1}{z} - \frac{p}{\mathcal{C}(z)}\Big(\mathcal{N}_0(\sigma) - \mathcal{N}_{d_{uv}}(\sigma)\Big) \\
&= -\frac{1}{z} - \frac{p}{\mathcal{C}(z)}\Big(1 + \mathcal{T}_N(\sigma) - \mathcal{T}_{d_{uv}}(\sigma) - \mathcal{T}_{N-d_{uv}}(\sigma)\Big).
\end{aligned}} \tag{19}$$

and

$$\boxed{\det[\underline{H}(n,n_0;z)] = \det[\underline{H}(z)] + \frac{p}{\mathcal{C}(z)^2}\Big(\mathcal{N}_{d_{nu}}(\sigma) - \mathcal{N}_{d_{nv}}(\sigma)\Big)\,\mathcal{N}_{d_{un_0}}(\sigma).} \tag{20}$$

3 Part 3 — First-passage time (FPT) generating function

The FPT generating function from n_0 to n is

$$\tilde{F}_{n_0\to n}(z) = \frac{\tilde{S}_{n_0}(n,z)}{\tilde{S}_n(n,z)}. \tag{21}$$

Using (18) both in the numerator and with $n_0 = n$ in the denominator yields

$$\boxed{\tilde{F}_{n_0\to n}(z) = \frac{\tilde{Q}(d(n,n_0),z) - \dfrac{z\,p\Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,\tilde{Q}(d(u,n_0),z)}{1 + z\,p\Big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\Big)}}{\tilde{Q}(0,z) - \dfrac{z\,p\Big(\tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z)\Big)\,\tilde{Q}(d(u,n),z)}{1 + z\,p\Big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\Big)}}.} \tag{22}$$

3.1 Algebraic simplification and Chebyshev closed form

Starting from the nested-fraction form (22), define the two recurring algebraic blocks

$$D_Q(z) \equiv 1 + zp\big(\tilde{Q}(0,z) - \tilde{Q}(d(u,v),z)\big), \qquad \Delta_n(z) \equiv \tilde{Q}(d(n,u),z) - \tilde{Q}(d(n,v),z). \tag{23}$$

Step 1: remove nested fractions. Multiply the numerator and denominator of (22) by $D_Q(z)$ to obtain the equivalent form without nested fractions:

$$\boxed{\tilde{F}_{n_0\to n}(z) = \frac{\tilde{Q}(d(n,n_0),z)\,D_Q(z) - zp\,\Delta_n(z)\,\tilde{Q}(d(u,n_0),z)}{\tilde{Q}(0,z)\,D_Q(z) - zp\,\Delta_n(z)\,\tilde{Q}(d(u,n),z)}.} \tag{24}$$

Step 2: substitute the Chebyshev packaging for $\tilde{Q}(d,z)$. Using $\sigma(z)$ from (4) and the Chebyshev form (10), define

$$\tilde{Q}(d,z) = \frac{\mathcal{N}_d(\sigma(z))}{\mathcal{C}(z)}, \qquad \mathcal{C}(z) \equiv qz\big(\sigma(z)^2 - 1\big)\mathcal{U}_{N-1}(\sigma(z)), \qquad \mathcal{N}_d(\sigma) \equiv \mathcal{T}_d(\sigma) + \mathcal{T}_{N-d}(\sigma).$$

Then

$$\Delta_n(z) = \frac{\mathcal{N}_{d(n,u)}(\sigma(z)) - \mathcal{N}_{d(n,v)}(\sigma(z))}{\mathcal{C}(z)}, \qquad D_Q(z) = \frac{\mathcal{C}(z) + zp\big(\mathcal{N}_0(\sigma(z)) - \mathcal{N}_{d(u,v)}(\sigma(z))\big)}{\mathcal{C}(z)}.$$

It is convenient to introduce the u–v dependent factor

$$\boxed{D(z) \equiv \mathcal{C}(z) + zp\big(\mathcal{N}_0(\sigma(z)) - \mathcal{N}_{d(u,v)}(\sigma(z))\big),} \tag{25}$$

so that $D_Q(z) = D(z)/\mathcal{C}(z)$.

Step 3: cancel common factors and obtain the closed form. Substituting the above expressions into (24) shows that both the numerator and denominator carry the same factor $\mathcal{C}(z)^{-2}$, which cancels. The result is the fully explicit closed form

$$\boxed{\tilde{F}_{n_0\to n}(z) = \frac{\mathcal{N}_{d(n,n_0)}(\sigma(z))\,D(z) - zp\,\Big(\mathcal{N}_{d(n,u)}(\sigma(z)) - \mathcal{N}_{d(n,v)}(\sigma(z))\Big)\,\mathcal{N}_{d(u,n_0)}(\sigma(z))}{\mathcal{N}_0(\sigma(z))\,D(z) - zp\,\Big(\mathcal{N}_{d(n,u)}(\sigma(z)) - \mathcal{N}_{d(n,v)}(\sigma(z))\Big)\,\mathcal{N}_{d(u,n)}(\sigma(z))}.} \tag{26}$$

Here $\mathcal{N}_0(\sigma) = \mathcal{T}_0(\sigma) + \mathcal{T}_N(\sigma) = 1 + \mathcal{T}_N(\sigma)$.

Remark (antipodal distance for even N). If N is even and some distance equals $d = N/2$, then $\mathcal{N}_{N/2}(\sigma) = 2\,\mathcal{T}_{N/2}(\sigma)$, and the above formulas simplify accordingly.

Numerical validation (double precision)

We validated (10), (18), and (26) against direct matrix computation of the resolvent. For each parameter set (N, q, p, u, v, z) we built the column-stochastic transition matrices $\underline{B}$ (ring) and $\underline{A}$ (directed shortcut), and computed

$$G_B(z) = (\underline{I} - z\underline{B})^{-1}, \qquad G_A(z) = (\underline{I} - z\underline{A})^{-1}.$$

The ground-truth generating functions are $\tilde{Q}_{n_0}(n, z) = [G_B(z)]_{n,n_0}$ and $\tilde{S}_{n_0}(n, z) = [G_A(z)]_{n,n_0}$. For first passage we used $\tilde{F}_{n_0 \to n}(z) = [G_A(z)]_{n,n_0}/[G_A(z)]_{n,n}$ for $n_0 \neq n$. We report the maximum absolute error over all indices:

$$\max_{n,n_0} |\Delta\tilde{Q}|, \qquad \max_{n,n_0} |\Delta\tilde{S}|, \qquad \max_{n_0 \neq n} |\Delta\tilde{F}|.$$

N	q	p	u	v	z	$\max\lvert\Delta\tilde{Q}\rvert$	$\max\lvert\Delta\tilde{S}\rvert$	$\max\lvert\Delta\tilde{F}\rvert$
9	0.6	0.2	1	4	0.3	2.22e-16	4.44e-16	5.55e-17
10	0.7	0.1	2	7	0.5	4.44e-16	2.22e-16	5.55e-17
12	0.5	0.3	0	6	0.6	4.44e-16	4.44e-16	8.33e-17
15	0.8	0.05	3	4	0.4	2.22e-16	4.44e-16	1.11e-16
20	0.4	0.2	5	13	0.75	1.33e-15	1.33e-15	1.67e-16

In a sweep of 200 random cases ($N \leq 40$, seed $= 0$, $q \in [0.2, 0.9]$, $z \in [0.2, 0.85]$, $p \in [0, 1-q]$), we found

$$\max|\Delta\tilde{Q}| = 3.11 \times 10^{-15}, \qquad \max|\Delta\tilde{S}| = 3.11 \times 10^{-15}, \qquad \max|\Delta\tilde{F}| = 4.44 \times 10^{-16}.$$

These residuals are consistent with floating-point round-off and the conditioning of matrix inversion as $z \to 1^-$.