2D Blackboard-Style Case: Start/Target at Wall Endpoints

Summary

- **Start/target at the corridor endpoints**: $S = (30, 45)$ and $T = (30, 15)$, with the same two-wall corridor geometry.
- **No bimodality**: $t_{p1} \approx 102$, valley ratio $= 1$ (single peak + long tail); absorption probability ≈ 0.604 at $t_{\max} = 3000$.
- **Small N vertical rectangle also fails**: on an 18×30 rectangle with the same endpoint placement, $t_{p1} \approx 40$, valley ratio $= 1$, absorption ≈ 0.664 at $t_{\max} = 2000$.
- **Scan conclusion**: changing g_y, the opening-arrow strength, or adding only an outer sticky belt does not create a second peak; a late peak appears only after adding an *outer sticky + directional loop* (an added slow channel).
- **Full figure set included**: environment, three heatmap frames, fast/slow path density, FPT curve + zoom, diagnostic plot, and channel decomposition.

Global parameters

case	N	q	g_x	g_y	$t_{\max}$	(start, target)
Z	60	0.8	0.00	-0.20	3000	(30,45)→(30,15)
S	30	0.8	0.00	-0.20	2000	(9,22)→(9,9)

Table 1: Global parameters for the wall-endpoint case.

Statistics

case	method	t_{p1}	t_{p2}	valley ratio	absorption	truncated (MC)	P_{fast} (MC)
Z	windowed	102	1500	1.000	0.604	0.395	0.540
S	windowed	40	1000	1.000	0.664	0.333	0.543

Table 2: Statistics for the wall-endpoint case. t_{p2} is the late-window maximum (not a genuine second peak).

Numerical scans (based on case Z)

Scan takeaways Z0–Z4 remain single-peak + long tail; the late-window "peak" is a window-edge artifact. Adding only a sticky belt does not induce bimodality; a clear second peak appears only when an outer directional loop is introduced (Z5/Z6), which effectively creates a new slow channel.

variant	modification	absorption	t_{p1}	t_{p2}	valley ratio	bimodal
Z0	base ($g_y = -0.2,\ \delta_{\text{open}} = 0.6$)	0.604	102	1500	1.000	no
Z1	$g_y = 0$	0.696	137	1500	1.000	no
Z2	$g_y = -0.4$	0.573	81	1500	1.000	no
Z3	$\delta_{\text{open}} = 0.2$	0.611	102	1500	1.000	no
Z4	outer sticky belt ($\alpha = 0.2$)	0.603	102	1500	1.000	no
Z5	sticky ($\alpha = 0.2$) + clockwise loop bias ($\delta = 0.1$)	0.603	102	2996	0.300	yes
Z6	stronger sticky ($\alpha = 0.05$) + clockwise loop bias ($\delta = 0.1$)	0.623	102	1990	0.014	yes

Table 3: Scan results at $t_{\max} = 3000$. For Z0–Z4, t_{p2} is the late-window maximum (not a genuine second peak).

1 Model definition and coordinates

The state space is $\Omega \subseteq \{1, \dots, N_x\} \times \{1, \dots, N_y\}$ (rectangles are allowed), and the target (x_t, y_t) is absorbing. One-step probabilities are

$$p_{\text{left}} = \frac{q}{4}(1 + g_x), \quad p_{\text{right}} = \frac{q}{4}(1 - g_x), \tag{1}$$

$$p_{\text{down}} = \frac{q}{4}(1 + g_y), \quad p_{\text{up}} = \frac{q}{4}(1 - g_y), \tag{2}$$

$$p_{\text{stay}} = 1 - q. \tag{3}$$

We use the convention "y increases downward," so $g_y > 0$ biases downward and $g_x > 0$ biases leftward. Both the outer boundary and internal walls are **fully reflecting**: any attempted move outside Ω or across a wall is reassigned to p_{stay}.

Local heterogeneity (composition rules)

- local bias (arrow): multiply p_{stay} by $(1 - \delta)$ and add $\Delta = \delta\, p_{\text{stay}}$ to the arrow direction.
- sticky: set $q_u = \alpha(u)q$ so $p_{\text{stay}}(u) = 1 - q_u$ is larger, then apply local bias.
- barrier: remove edge $u \leftrightarrow v$ and add its probability to $p_{\text{stay}}(u)$.
- door: pass with probability p_{pass}, reflect the remainder to $p_{\text{stay}}(u)$.

2 Exact FPT recursion

Define $\tau = \inf\{t \geq 1 :\ X_t = (x_t, y_t)\}$, $f(t) = \Pr(\tau = t)$. Let Q be the transient submatrix and r the one-step flow into the absorbing state, then

$$f(t) = r^\top Q^{t-1} p_0, \qquad t \geq 1,$$

with the equivalent recursion $p_{t+1} = Q p_t$, $f(t) = r^\top p_t$. Exact recursion is the baseline; AW inversion and MC are used for validation and channel statistics.

3 Figure legend

Figure 1 shows the unified symbols: red square = start, blue diamond = target; solid thick lines = reflecting walls, dashed lines = doors, translucent shading = sticky region; red arrows = local bias; the outer arrow gives the global bias (g_x, g_y). Channel cartoon is in Fig. 2.

4 Case Z: Start/target at wall endpoints

Geometry and parameters

- Internal walls: $x = 28 \leftrightarrow 29$ and $x = 31 \leftrightarrow 32$ for $y = 15..45$.
- Corridor bias: upward arrows on $x = 29..31$; core segment $y = 15..45$ uses $\delta_{\text{core}} = 0.95$, open segments $y = 10..14, 46..50$ use $\delta_{\text{open}} = 0.6$.
- Start/target: $S = (30, 45),\ T = (30, 15)$; $g = (0, -0.2)$, $t_{\max} = 3000$.



Figure 1: Symbol legend.

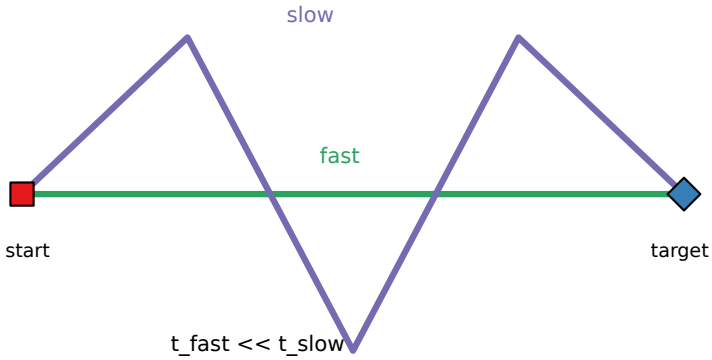
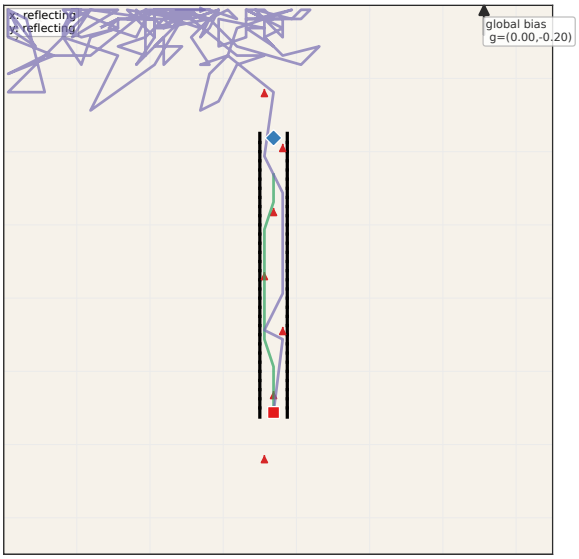


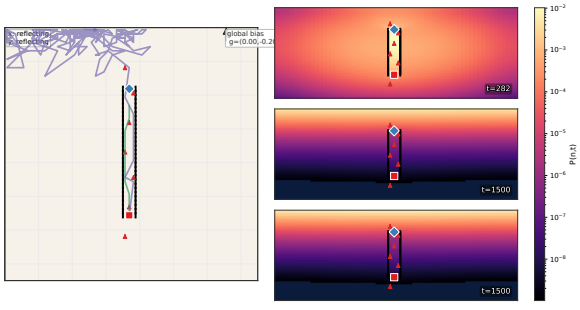
Figure 2: Channel cartoon (fast/slow mixture).

Outcome Early peak at $t \approx 102$, valley ratio $= 1$; absorption probability is ≈ 0.604 at $t_{\max} = 3000$, so no stable late peak appears.

Why there is no bimodality With both endpoints inside the corridor, the strong upward corridor bias defines the dominant drift timescale. Trajectories that exit the corridor do not enter a *quasi-closed* slow channel; they diffuse broadly in the outer domain and eventually return, producing a long tail rather than a concentrated late-time peak. In mixture terms, the "slow" conditional distribution is too broad to carve out a valley and a second peak in $f(t)$.

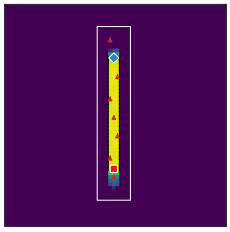


(Z1) Environment

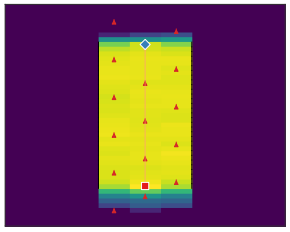


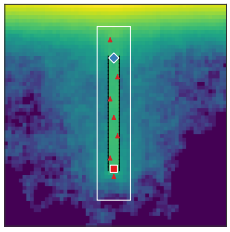
(Z2) Heatmaps

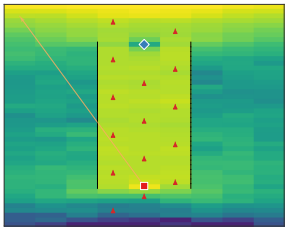
Figure 3: Case Z environment and $P(n, t)$ heatmaps.



(Z3) Fast-path density

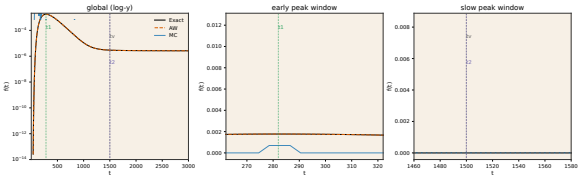




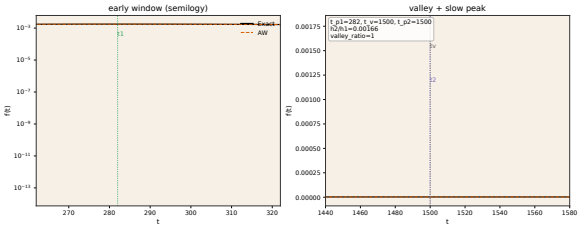


(Z4) Slow-path density

Figure 4: Case Z fast/slow path densities.

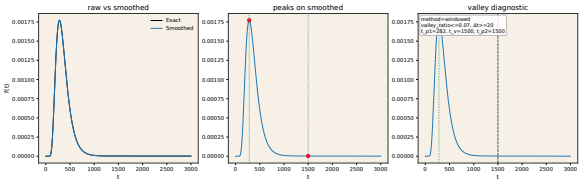


(Z5) FPT curve

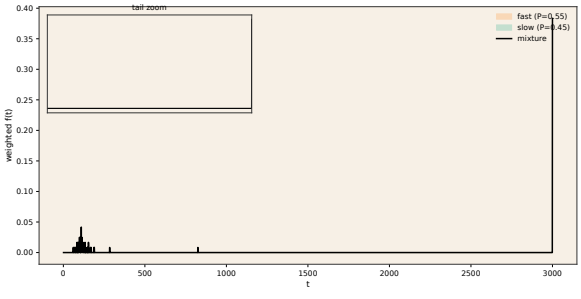


(Z6) Peak/valley zoom

Figure 5: Case Z FPT and zoomed view.



(Z7) Diagnostic



(Z8) Channel decomposition

Figure 6: Case Z diagnostics and channel decomposition.

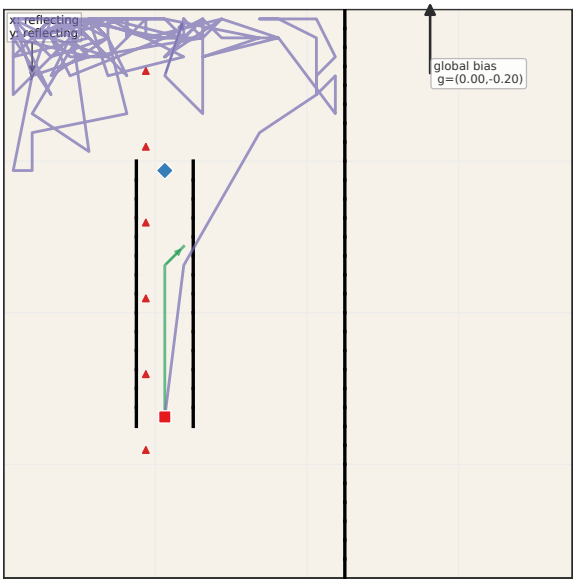
5 Case S: Small vertical rectangle (endpoints on walls)

Geometry and parameters

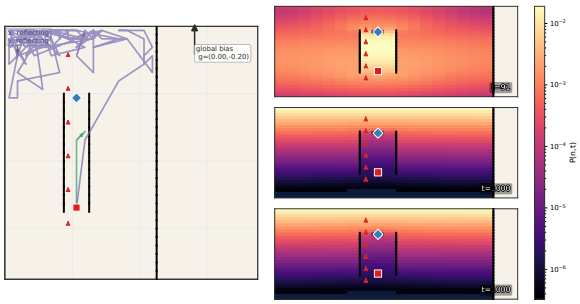
- Domain: 18×30 rectangle (allowed set $x = 1..18,\ y = 1..30$ embedded in an $N = 30$ grid).
- Internal walls: $x = 7 \leftrightarrow 8$ and $x = 10 \leftrightarrow 11$ for $y = 9..22$.
- Corridor bias: upward arrows on $x = 8..10$; core segment $y = 9..22$ uses $\delta_{\text{core}} = 0.95$, open segments $y = 4..8, 23..27$ use $\delta_{\text{open}} = 0.6$.
- Start/target: $S = (9, 22),\ T = (9, 9)$; $g = (0, -0.2)$, $t_{\max} = 2000$.

Outcome Early peak at $t \approx 40$, valley ratio $= 1$; absorption probability is ≈ 0.664 at $t_{\max} = 2000$, so no stable late peak appears.

Mechanism Shrinking to a vertical rectangle does not create a slow channel. Excursions outside the corridor diffuse broadly and return, producing a long tail rather than a concentrated late-time peak.

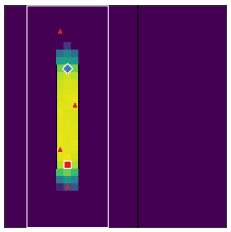


(S1) Environment

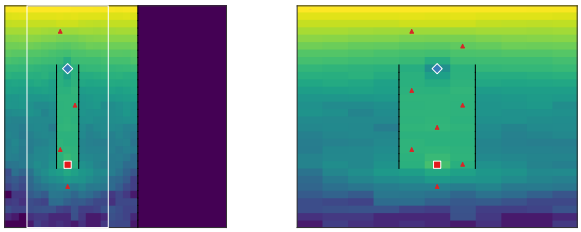


(S2) Heatmap frames

Figure 7: Case S environment and $P(n, t)$ heatmaps.



(S3) Fast-path density

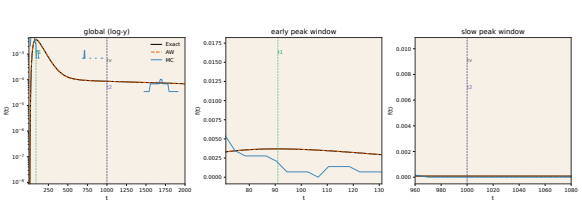


(S4) Slow-path density

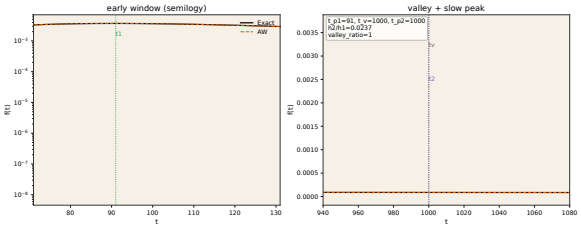
Figure 8: Case S fast/slow path densities.

Data and reproduction

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python3 code/z_scan.py
```

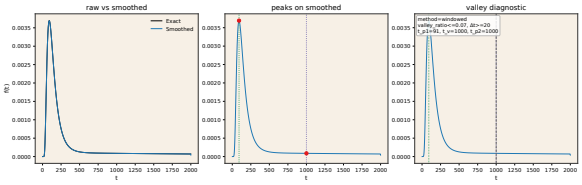


(S5) Multiscale FPT

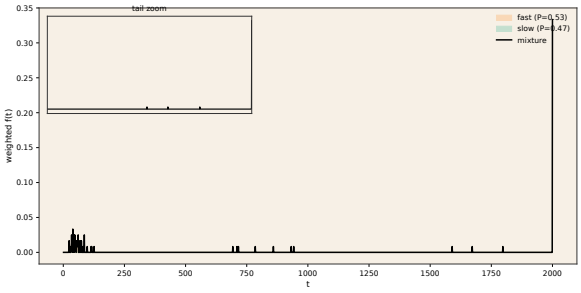


(S6) Peak/valley zoom

Figure 9: Case S FPT curve and zoom.



(S7) Diagnostic



(S8) Channel decomposition

Figure 10: Case S diagnostics and channel decomposition.

```
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↪  grid2d_blackboard_bimodality_en.tex
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