# 1D Ring and 2D Two-Target Bimodality Brief

Focus: what was scanned, where double peaks appear, and which criteria should be used.

## Final Conclusion

**Overall conclusion.** Across the current 1D and 2D tests, a visible double peak is best understood as competition between two first-passage arrival budgets:

$$\text{visible double peak} \approx \text{channel mass balance} + \text{timescale separation} + \text{resolved valley}.$$

The strongest conclusion is therefore not that one parameter alone causes double peaks. The robust conclusion is that double peaks appear when two arrival channels both carry visible probability mass, arrive on separated timescales, and leave a valley satisfying the R3 visual rule $\rho = h_v / \min(h_1, h_2) \leq 0.8$. Here, *channel mass balance* means that neither arrival channel is negligible: the early channel must be high enough to form the first peak, but not so dominant that it erases the delayed peak. *Timescale separation* means that the two channels reach the target on sufficiently different typical time ranges, so their peaks do not merge into a single broad maximum.

**1D conclusion.** The 1D shortcut ring gives the cleanest mechanism. The shortcut creates the early arrival budget and the non-shortcut paths create the delayed budget. The dominant controls are $N$, $\beta$, $K$, and shortcut/target placement. $N$ controls whether the two timescales can separate at all. $\beta$ controls the mass balance: too small gives no shortcut peak, while too large suppresses the delayed non-shortcut peak. $K = 4$ is more robust than $K = 2$ because overshoot or jump-over path classes keep delayed mass visible.

**Two-target conclusion.** The two-target scans show that a shortcut is not necessary for double peaks. Target-channel competition can produce classifier-confirmed `double_peak` cases when one target supplies an early channel and the other supplies a delayed channel. The central object is therefore the channel decomposition, not the shortcut itself.

**2D conclusion.** The 2D scans support the same channel-competition mechanism, but the raw curves are less naturally smooth because small-grid, discrete-time paths to a near target create path-length and parity oscillations. The best current 2D design is a local-bias basin: global bias keeps the far-target channel alive, weak local inward bias captures a near-target channel, and moderate local holding broadens the near-channel residence. Strong local attraction alone is not the best mechanism because it makes near absorption too fast and sharp. This is a promising pilot mechanism rather than a complete 2D phase diagram.

**Most influential factors.** The practical ranking from the current evidence is:

1. channel mass balance,
2. timescale separation,
3. target/shortcut geometry,
4. bias strength ($\beta$ in 1D, $g$ in 2D),
5. local residence/holding $h$ in 2D,
6. jump range $K$,
7. visual/classifier threshold.

The first two factors are the real bottleneck. The remaining parameters matter because they tune either the mass in each channel or the separation between their arrival times.

## One-Page Readout

**Main message.** The current evidence supports two routes to a visible double peak. In the 1D shortcut ring, the distribution splits into a fast shortcut-arrival budget and a slow non-shortcut budget. In the two-target problems, the split is a target-channel competition: an early near/shortcut-favoured channel and a delayed far/non-shortcut channel. A curve should be called a visual double peak only when the two budgets are both present, separated in time, and separated by a valley that is low relative to the smaller peak.

**Criterion used in this brief.** The visual criterion is aligned with the fixed-shortcut report. Let $t_1 < t_2$ be the two filtered local-maximum times, $h_i = f(t_i)$, and let $h_v$ be the intervening valley height. Define

$$\rho = \frac{h_v}{\min(h_1, h_2)}.$$

R1 means two filtered local maxima with enough time separation and balanced heights; R2 is a permissive dip test; R3 is the main visual double-peak rule $\rho \leq 0.8$; R4 is the stronger visual rule $\rho \leq 0.6$. The important point is that the visual rule uses $h_v/\min(h_1, h_2)$, not $h_v/\max(h_1, h_2)$, because the valley must be below the smaller peak to look like two resolved peaks.

| Problem | Evidence in this brief | Mechanism | Practical conclusion |
|---|---|---|---|
| 1D shortcut ring | $\beta$- and $N$-scans; reference curves at $\beta = 0.01$ | $\beta$ creates a fast shortcut budget; $N$ controls separation from the slow non-shortcut budget | There is a middle window: no shortcut gives no early peak; too much shortcut mass suppresses the late peak. |
| 1D two-target ring | 240-case scan; 34 classifier-confirmed `double_peak` cases | Two absorbing targets compete as early and delayed channels | The shortcut is helpful but not required; channel balance and drift matter. |
| 2D baseline two-target scan | 20-case fixed-angle heatmap; 7 labelled `double_peak` cases | Near/far target competition under global east bias | The mechanism exists, but the visible shape can be jagged or weak on a small grid. |
| 2D local-bias basin pilot | 576-case pilot; 210 raw labels, 126 with actual local inward bias | Weak local capture plus local holding broadens the near-channel budget while global bias keeps the far channel alive | Best smooth examples use weak local attraction and moderate residence, not strong capture alone. |

Table 1: Summary: phenomenon, evidence, mechanism, and takeaway.

**Most important parameter levers.** For 1D, the decisive parameters are $N$, $\beta$, $K$, shortcut placement, target placement, and drift. For 2D, the decisive parameters are near-target position, global bias $g$, basin radius $R_b$, local inward bias $\ell$, and local hold $h$. The useful regime is not simply "larger bias". It is a balance: enough early-channel mass to make the first peak, enough delayed-channel mass to keep the second peak, and enough residence or path-length spread to avoid a narrow spike.

## 1D Shortcut Ring

**Reference configuration.** Ring sites are labelled $1, \ldots, N$. The reference case uses $N = 100$, start $n_0 = 1$, absorbing target 51, shortcut source $6 = n_0 + 5$, and shortcut destination $52 = 51 + 1$. Thus the shortcut is one-way, $6 \to 52$. Unless stated otherwise, $q = 2/3$, $\rho = 1$, lazy_selfloop, and the displayed reference case uses $\beta = 0.01$.

## What Was Scanned

**Operational definitions.** In the figures and scans below, a channel is a labelled source of first-passage probability, such as shortcut versus non-shortcut paths, target 1 versus target 2, or near-target versus far-target arrivals. Channel mass balance asks how much probability each channel contributes around the relevant peak windows. Timescale separation asks whether the peak windows of those channels are far enough apart to create a valley. The quantitative visual check is then the R3 ratio $\rho = h_v / \min(h_1, h_2) \leq 0.8$.

| Scan | Parameters varied | Observed double-peak window | Readout |
|---|---|---|---|
| Shortcut ring, fixed $N = 100$ | $\beta$ for $K = 2, 4$; $q = 2/3$, $\rho = 1$, $6 \to 52$ | $K = 2$ : $\beta = 0.0005$–$0.05$; $K = 4$ : $\beta = 0.001$–$0.15$ | $\beta = 0$ gives no shortcut peak. Too-large $\beta$ collapses the shape toward an early shortcut-dominated peak. |
| Shortcut ring, fixed $\beta = 0.02$ | $N = 10, 12, \ldots, 200$, $K = 2, 4$, antipodal target and shortcut just beyond target | $K = 2$ : $N \geq 58$; $K = 4$ : $N \geq 72$ | Small rings do not separate the fast and slow timescales enough. Larger $N$ creates the late non-shortcut peak. |
| Shortcut ring, fixed $\beta = 0.01$ | $N = 50, 52, \ldots, 300$, $K = 2, 4$ | $K = 2$ : $N \geq 60$; $K = 4$ : $N \geq 74$ | The practical minimum size is roughly $N \approx 60$ for $K = 2$ and $N \approx 70$–$75$ for $K = 4$. |
| Two-target 1D ring | $L = 31, 51$, $K = 2$, drift $g \in \{-0.6, -0.3, 0, 0.3, 0.6\}$, $\beta \in \{0, 0.03\}$, starts and two target locations | 34 classifier-confirmed `double_peak` cases out of 240; 14 without shortcut, 20 with shortcut | Double peaks can occur from target-channel competition alone. The shortcut is helpful but not necessary. |
| Two-target $N$-$\beta$ scan | $N = 30, 35, \ldots, 80$, $\beta = 0, 0.01, \ldots, 0.08$, $K = 2, 4$ | $K = 2$ : $N = 45$–$80$, $\beta = 0.01$–$0.08$; $K = 4$ : $N = 50$–$80$, $\beta = 0.01$–$0.08$ | The two-target mechanism needs enough spatial separation and a channel split; otherwise the result is unimodal, shoulder, or local bump. |

Table 2: Existing 1D ring scans already present in the repository.

## 1D Ring Parameters That Decide the Shape

**Best format here: parameter cards, not a wide table.** The important point is not only what was scanned, but which failure mode each parameter controls. A wide table becomes hard to read, so the report states each parameter as a small decision rule.

**$N$ or $L$** Controls timescale separation. If the ring is too small, fast and slow arrival budgets merge into one peak. In the shortcut scan, visually useful cases start around $N \approx 60$ for $K = 2$ and $N \approx 70$–$75$ for $K = 4$.

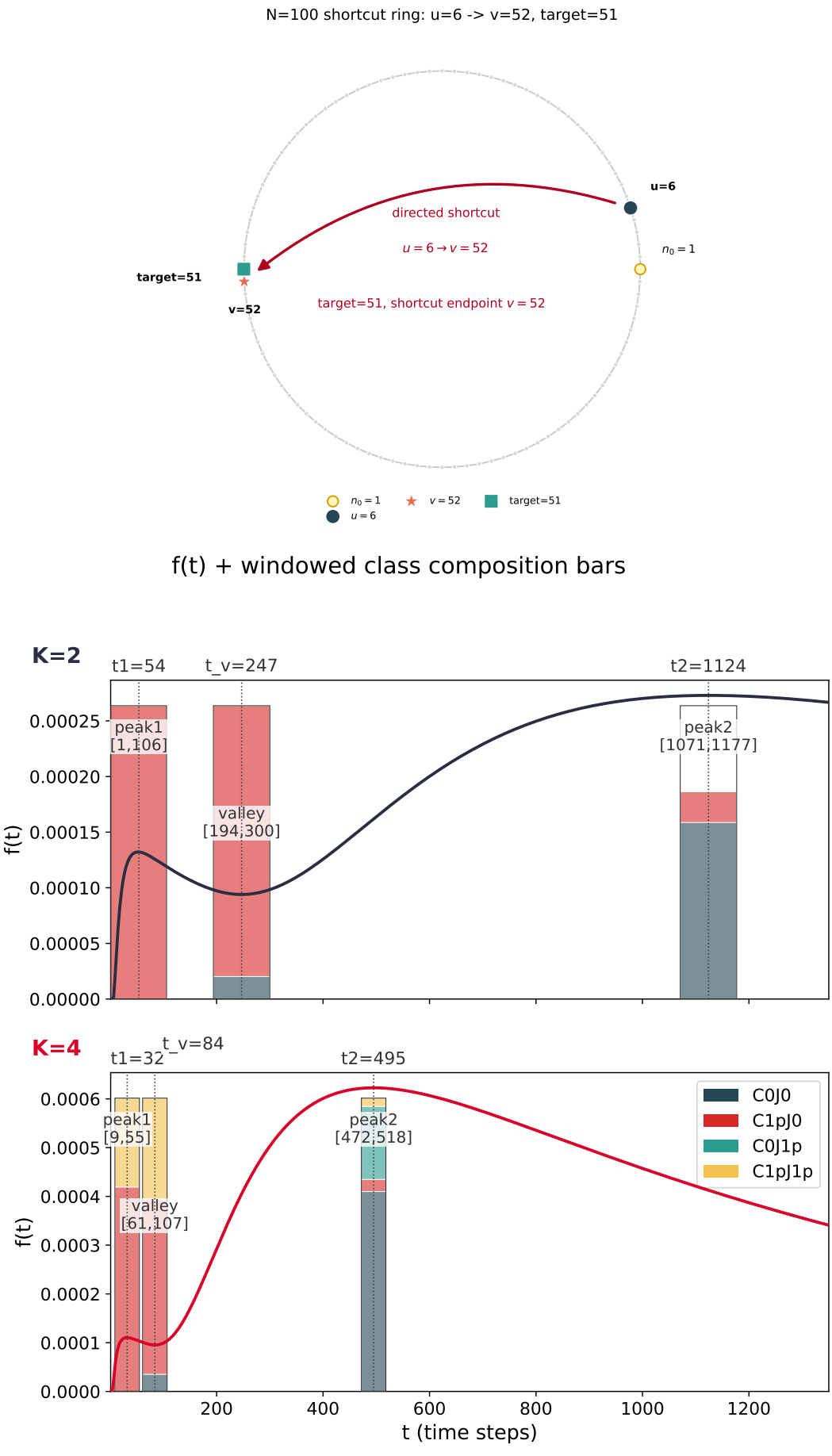



Figure 1: Top: reference ring geometry. Bottom: first-passage curves at $\beta = 0.01$. The early peak is shortcut-driven. The late peak is the slow non-shortcut budget; for $K = 4$, jump-over/overshoot trajectories add a delayed component.

$\beta$ Controls how much mass enters the fast shortcut channel. If $\beta = 0$, the shortcut-driven early peak is absent. If $\beta$ is too large, shortcut arrivals dominate and the late non-shortcut peak becomes a shoulder or disappears. At $N = 100$, the useful window is about 0.0005–0.05 for $K = 2$, and 0.001–0.15 for $K = 4$.

$K$ Controls path classes and overshoot/jump-over delays. $K = 4$ keeps a wider $\beta$ window than $K = 2$ because delayed non-shortcut or overshoot trajectories remain visible. This is a mechanism statement, so $K$ should always be interpreted together with target and shortcut geometry.

**Shortcut placement**
Controls whether the shortcut creates a separate early arrival budget. If it does not route mass into the target region, there is no early shortcut peak. If it routes mass too efficiently into the absorbing region, the early channel overwhelms the late channel. The reference $6 \to 52$ with target 51 creates an early budget without making every trajectory immediate.

**Targets and start**
Control channel competition. If the targets are too similar in effective distance, they behave like one combined absorbing set. If one target dominates all mass, the other channel leaves only a shoulder. Double peaks occur when one target contributes an early channel and the other contributes a delayed channel.

**Drift** $g$ Tilts probability mass between target channels. With no useful tilt, the channel split may be weak. With strong drift, one channel is suppressed and the curve becomes a single peak or shoulder. Moderate drift can improve channel separation while keeping both channels visible.

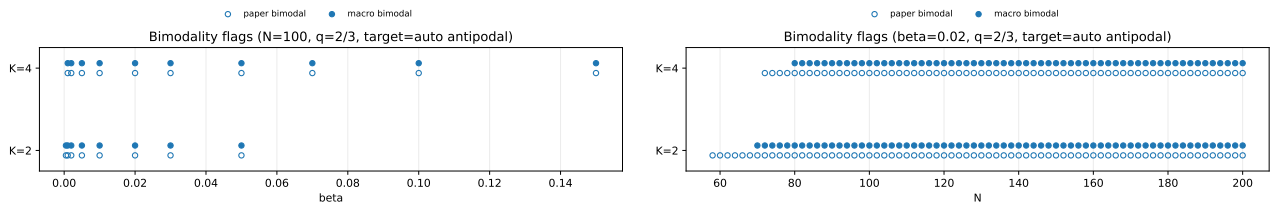



Figure 2: Left: $\beta$-scan at $N = 100$. Right: $N$-scan at $\beta = 0.02$. These figures show where the shortcut-ring classifier finds two separated peaks.

## 2D Baseline Near/Far Target Bias-Radius Scan

**Configuration.** This part has also been done, but only as a bounded $\theta = 0$ slice rather than a full angular scan. The grid is $15 \times 9$, the start is fixed at $(2, 4)$, the far target is fixed at $(12, 4)$, and the near target is placed east of the start at radius $r$. The boundary is reflecting attempted-outside-stays. The global east bias moves probability mass from stay to the east step:

$$p_E = q/4 + b, \qquad p_N = p_S = p_W = q/4, \qquad p_{\text{stay}} = 1 - q - b,$$

with $q = 0.7$. The scan uses $r = 2, 3, 4, 5$ and $b = 0, 0.04, 0.08, 0.12, 0.16$, giving 20 cases.

The old radius-scan schematic is not used as the main configuration figure in this brief. It is kept only as a baseline heatmap: a fixed-angle $\theta = 0$ scan of near-target radius and global bias. The main 2D design used below is the local-bias basin configuration.

**Classifier-confirmed 2D double-peak cases.** The scan found 7 `double_peak` cases out of 20:

$$(r, b) = (2, 0.16),\ (3, 0),\ (3, 0.16),\ (4, 0.08),\ (4, 0.12),\ (5, 0.12),\ (5, 0.16).$$

| Scan | Parameters varied | Readout |
|---|---|---|
| 2D near/far target, $\theta = 0$ | $r = 2, 3, 4, 5$, $b = 0, 0.04, 0.08, 0.12, 0.16$; start $(2, 4)$, far $(12, 4)$ | Double peaks appear when near-target hits and delayed far-target hits both retain visible mass. Intermediate/strong east bias often helps, but distance alone is not sufficient. |

Table 3: Existing 2D bias-radius scan. Positive claims require the stored classifier label to be exactly `double_peak`.

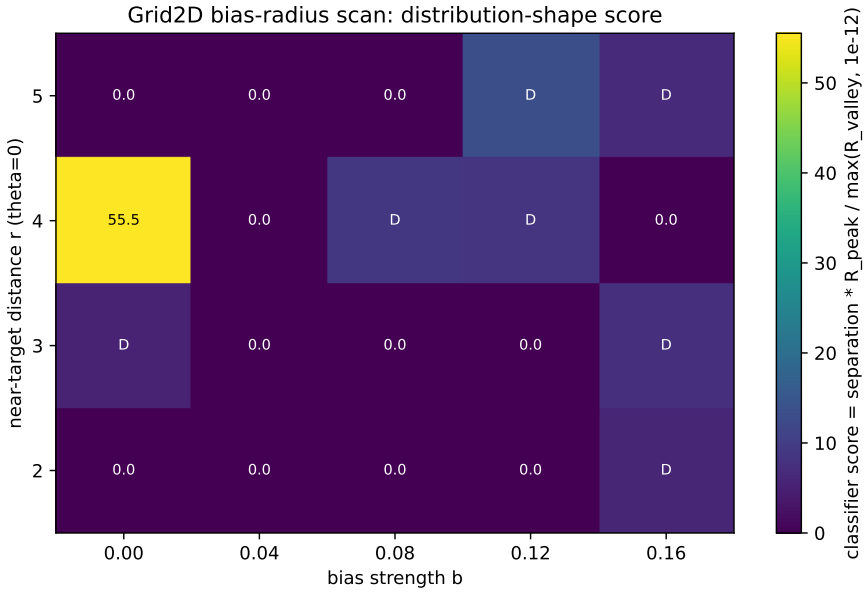



Figure 3: 2D heatmap for the fixed $\theta = 0$ slice. Color is the classifier score, not by itself a double-peak claim. The actual positive cases are the seven rows labelled `double_peak` in the metrics table.

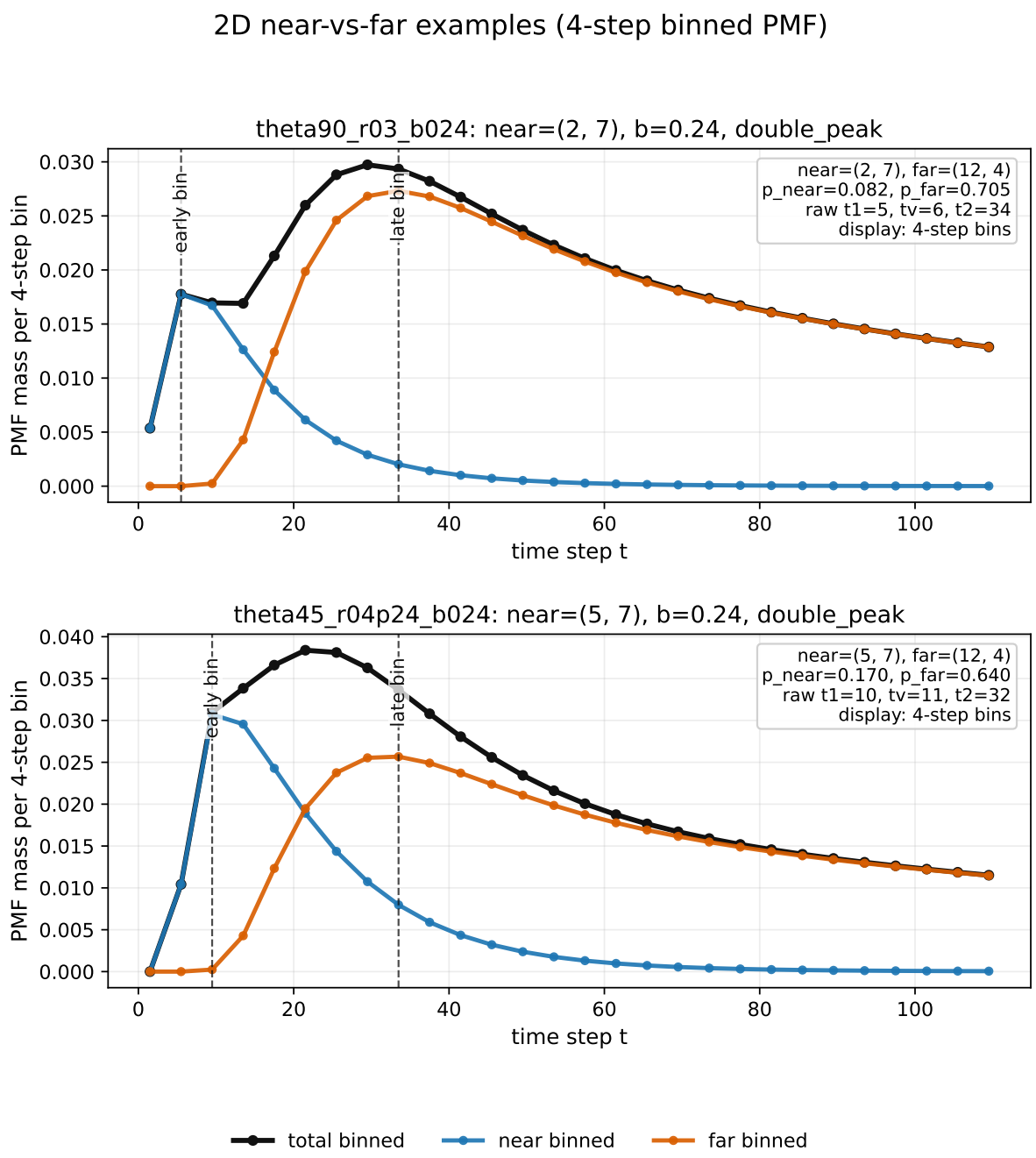



Figure 4: Two clearer 2D double-peak examples from a targeted $\theta$-expanded search, displayed as 4-step binned probability mass for readability. The classifier labels and peak times still come from the raw exact discrete-time PMF. In both cases the near target is off the east-bias line, so the early peak is a near-target channel and the late peak is dominated by delayed arrival at the far target. These examples are illustrative, not a full angular phase diagram.

**Why the display is binned.** The exact computation gives a raw discrete-time PMF on a small grid, so the raw curves contain parity and path-length oscillations, especially near the close target. To avoid a jagged presentation figure, Figure 5 bins probability mass into 4-step windows:

$$F_k = \sum_{t=4k}^{4k+3} f(t).$$

This preserves probability mass and is only a display transform. The scientific claims still use the raw classifier, raw peak times, and near/far channel decomposition.

## 2D Main Configuration: Local-Bias Basin

**Question.** The fixed global-bias two-target scan can produce classifier-confirmed `double_peak` cases, but the visual shape is often weak because the near target is hit by a small number of short paths. This motivates testing a plain 2D rectangle with a local bias basin around the near target. The aim is to create a broader near-target residence budget while keeping a delayed far-target budget alive.

**Configuration.** The grid is a reflecting $23 \times 13$ rectangle. The start is $(3, 6)$, the far target is $(19, 6)$, and the far target is aligned with a global east bias. The near target is moved off the east-bias line, e.g. $(5, 9)$ or $(5, 3)$. Around the near target, a local basin of radius $R_b$ is added. Inside this basin there are two local effects:

1. a weak inward bias pointing toward the near target;
2. a local holding/slowing term that increases residence in the basin.

Absorbing targets still take precedence: once either target is hit, the process stops.

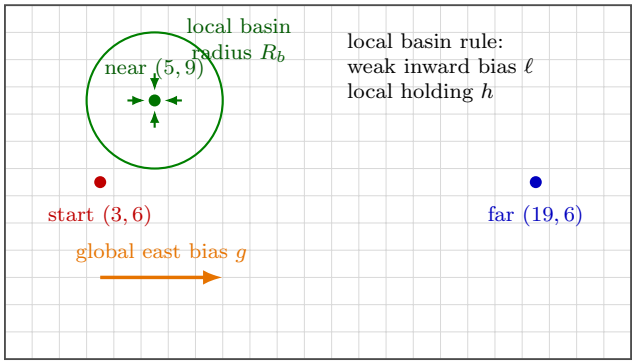



Figure 5: Configuration for the local-bias basin test. The geometry remains a plain reflecting rectangle. The added mechanism is a local near-target basin.

**Small scan.** The first local-bias scan used:

$$R_b \in \{2.5, 3.5, 4.5\}, \quad g \in \{0.08, 0.12, 0.16, 0.20\}, \quad \ell \in \{0, 0.04, 0.08, 0.12\}, \quad h \in \{0, 0.25, 0.45\}.$$

The near targets scanned were $(5, 9), (7, 9), (5, 3), (7, 3)$. This gives 576 cases. The raw exact PMF classifier found 210 `double_peak` cases. Restricting to actual local inward bias $\ell > 0$, it found 126 `double_peak` cases.

**2D parameter cards.** The 2D result is more sensitive to presentation because the raw exact PMF on a small grid has path-length oscillations. The parameters below should therefore be read as mechanism levers, with the raw classifier retained for formal labels and binned plots used only for readability.

**Global bias $g$**

Scanned $0.08, 0.12, 0.16, 0.20$. This keeps the far-target channel alive and shifts mass toward delayed far arrival. If $g$ is too small, the far bump is weak. If $g$ is too large, the far route can dominate and reduce near/far balance. Representative readable cases use $g = 0.16$.

**Near position**
Scanned $(5, 9), (7, 9), (5, 3), (7, 3)$. This decides whether the near target captures a side channel or absorbs the main eastward stream. Readable examples place the near target off the main east-bias line, especially $(5, 9)$ or $(5, 3)$.

$R_b$ Scanned $2.5, 3.5, 4.5$. This sets the spatial width of local capture and residence. Too small behaves like a short-path spike; too broad can flatten the near/far distinction. The clearest local-bias-positive example uses $R_b = 2.5$.

$\ell$ Scanned $0, 0.04, 0.08, 0.12$. This creates inward drift toward the near target. Too small gives little basin effect; too large makes near absorption too fast and sharp. The useful displayed value is weak, $\ell = 0.04$.

$h$ Scanned $0, 0.25, 0.45$. This is the main smoothing lever because it broadens residence inside the near basin. Too small leaves a jagged short-path contribution; too large can make the near-channel timing too diffuse. The clearest displayed value is $h = 0.25$.

**2D parameter conclusion.** The useful setting is a balance: weak local capture plus moderate residence broadens the near channel, while global bias maintains the delayed far channel. Stronger local attraction alone is not the answer.

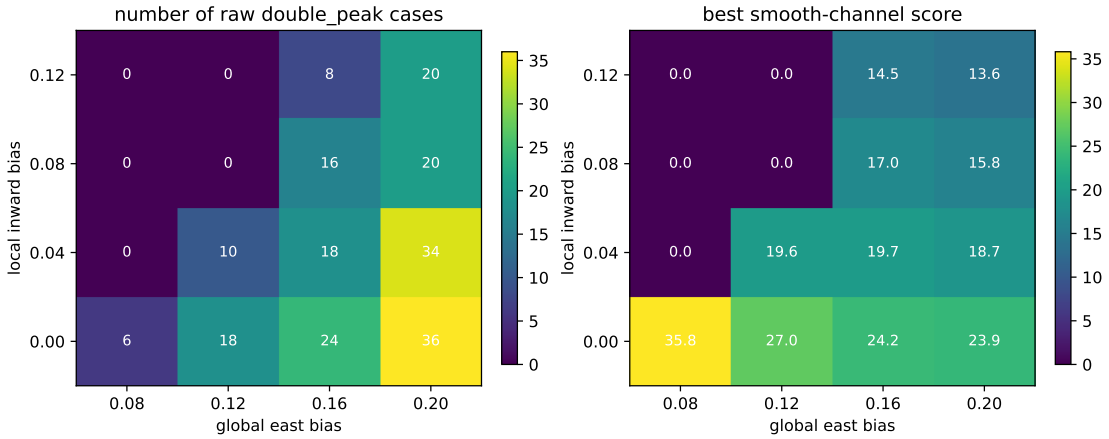



Figure 6: Local-bias basin scan. Left: number of raw classifier-confirmed `double_peak` cases aggregated over near-target position, basin radius, and holding strength. Right: best smooth-channel score used to choose readable examples.

**Preliminary conclusion.** The useful mechanism is not strong local attraction by itself. Strong local bias tends to make near absorption too fast and sharp. The more promising design is a weak local capture basin plus moderate local residence: global east bias keeps the far-target channel alive, while the local basin broadens the near-target contribution. This produces smoother two-target double-peak-like curves. This is still a pilot scan, not a full 2D local-bias phase diagram.

## Double-Peak Criteria Aligned With the Fixed-Shortcut Report

**Do not use "two local maxima" alone.** That criterion is too permissive: it counts tiny numerical bumps and visually weak shoulders. The same R1–R4 language is used as in the fixed-shortcut report so that the two reports do not apply different standards to the same phenomenon.

Let $t_1 < t_2$ be the two filtered local-maximum times, $h_i = f(t_i)$, and let $h_v$ be the minimum value between the two peaks. Define

$$\rho = \frac{h_v}{\min(h_1, h_2)}.$$

**R1** **Two local maxima**: two filtered local peaks with enough time separation and balanced heights.

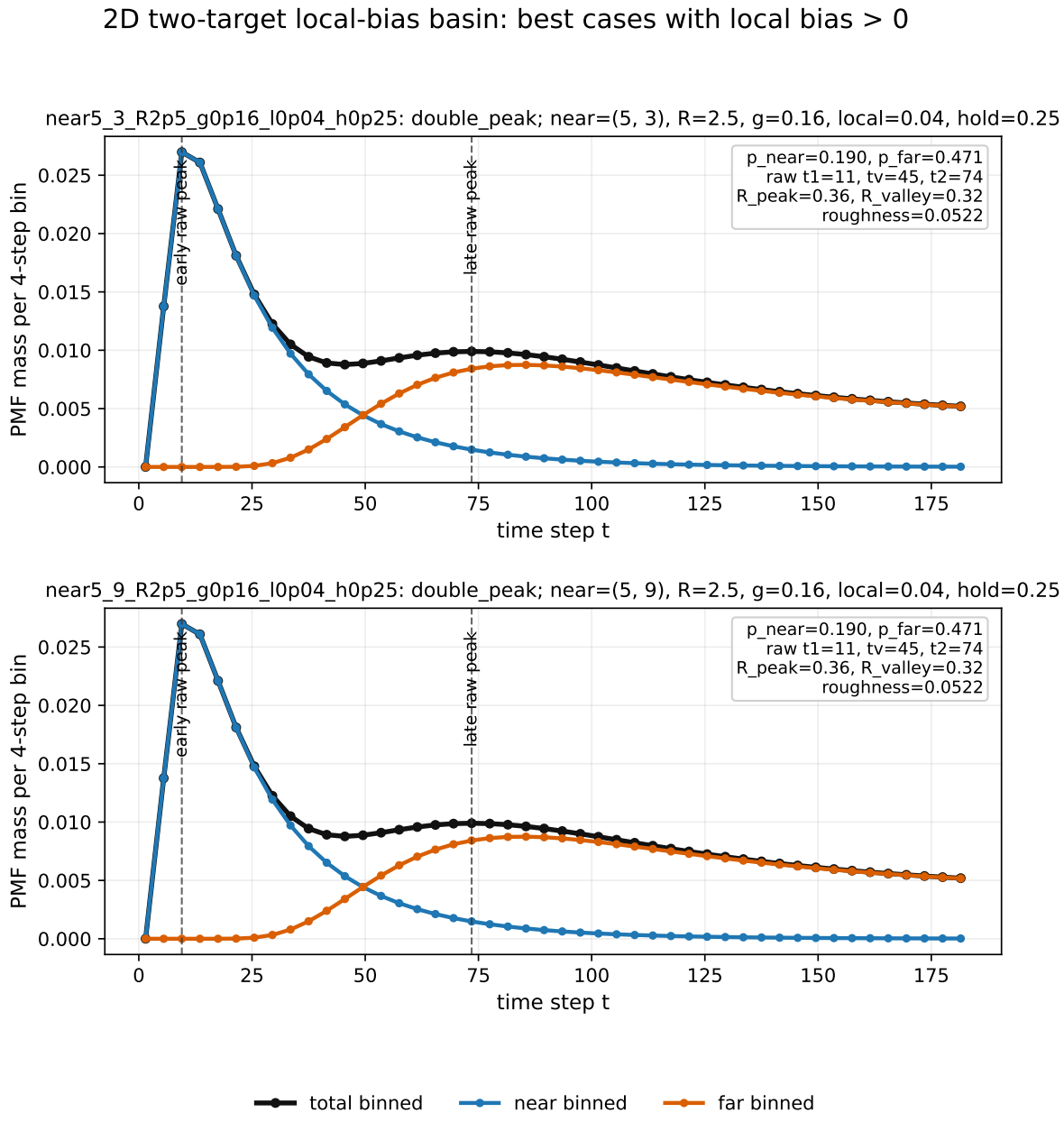



Figure 7: Best local-bias-positive examples, displayed as 4-step binned probability mass. These are still classified using the raw exact PMF. A representative setting is near $(5, 3)$ or $(5, 9)$, $R_b = 2.5$, global east bias $g = 0.16$, local inward bias $\ell = 0.04$, and local hold $h = 0.25$. The early peak is near-channel dominated, while the later broad bump is far-channel dominated.

**R2** **Permissive valley**: R1 plus $h_v/\max(h_1, h_2) \leq 0.9$. This only says that a dip exists.

**R3** **Visual double peak**: R1 plus $\rho = h_v/\min(h_1, h_2) \leq 0.8$. This is the main rule for saying that a curve visibly has two resolved peaks.

**R4** **Strong visual double peak**: R1 plus $\rho = h_v/\min(h_1, h_2) \leq 0.6$. This is reserved for strongly separated cases.

The reason for using $h_v/\min(h_1, h_2)$ in R3/R4 is simple: if the valley is close to the smaller peak, the curve may contain two mathematical local maxima but still look like a shoulder. Therefore, in this brief, `double_peak` labels from stored classifiers are treated as the formal data labels, while R3 is the preferred visual presentation rule. R2 is only a diagnostic for "there is some dip" and should not be used as the main visual claim.

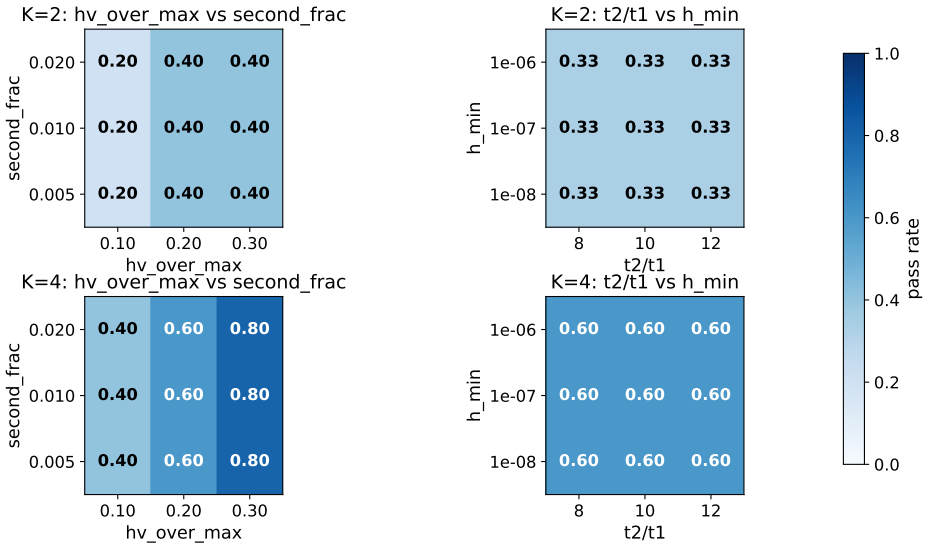



Figure 8: Threshold sensitivity scan. The scan varied the minimum peak height, second-peak fraction, $t_2/t_1$, and valley-depth threshold. This is why the report should distinguish candidate bimodality from visually clear bimodality.

## What Helps or Blocks Double Peaks

**Helps.** Double peaks are easiest to see when the system creates two arrival budgets with separated timescales. In the shortcut ring this is the fast shortcut budget versus the slow non-shortcut budget. In the two-target ring this is the near/early target channel versus the far/late target channel. In the 2D scan, the same logic becomes a near-target channel versus a far-target channel under eastward bias. In the new 2D local-bias design, global bias maintains the far channel, while weak local capture plus local holding broadens the near-channel budget.

**Blocks or weakens.** Too small $N$ compresses the two timescales into one peak. Too small $\beta$ removes the shortcut-driven early budget. Too large $\beta$ overweights the shortcut route and can erase the late component as a visible second peak. In the two-target scan, weak channel separation gives shoulder or local-bump labels rather than `double_peak`. In the 2D scan, $r = 2$ needs stronger bias to show a labelled double peak, while some larger-radius cases fail if the near and far channels do not both remain visible. The clearer near-vs-far examples found by the targeted $\theta$-expanded search place the near target away from the east-bias line, e.g. near $(2, 7)$ or $(5, 7)$ with $b = 0.24$. This lets the far target keep enough probability mass to form a delayed second peak.

**Role of $K$.** For the shortcut ring, $K = 4$ is more robust over $\beta$ than $K = 2$: the sampled bimodal interval expands from $\beta \leq 0.05$ for $K = 2$ to $\beta \leq 0.15$ for $K = 4$. The mechanism is that $K = 4$ has jump-over or overshoot classes that add a delayed contribution to the slow side of the distribution.

## Older Shortcut Convention Kept Separate

There is also an older shortcut convention for $K = 2$ with fixed shortcut $6 \to 56$ and target 56. That scan is useful as a threshold check, but it is not the same as the current reference configuration $6 \to 52$ with target 51. Under the older $6 \to 56$ convention, $\beta_c = 2q/((1-q)N) = 0.04$ is not an exact boundary for the existence of any two local maxima, but it works as a practical boundary for visually meaningful double peaks: above $\beta_c$, the valley becomes too shallow for a clear two-peak claim.

## Source Files

- Shortcut-ring beta and $N$ scans: `ring_lazy_jump_ext/artifacts/data/scan_beta_N100_K2.csv`, `scan_beta_N100_K4.csv`, `scan_N_K2_beta002.csv`, `scan_N_K4_beta002.csv`.
- Aligned shortcut-ring schematics: `final_multitimescale_fpt_encounter/code/plot_aligned_shortcut_ring_geometry.py` and `shortcut_ring_geometry_reference_6to52.pdf`.
- Larger $N$-scan at $\beta = 0.01$: `ring_lazy_jump_ext/artifacts/outputs/N_sweep_beta_selected/ N_sweep_metrics_beta0.01.csv`.
- Threshold sensitivity: `ring_lazy_jump_ext_rev2/artifacts/outputs/sensitivity/threshold_sweep.csv`.
- Two-target ring: `ring_two_target/artifacts/data/small_scan_metrics.csv`, `scan_bimodality_K2.csv`, and `scan_bimodality_K4.csv`.
- 2D two-target bias-radius scan: `grid2d_two_target_bias_radius/artifacts/data/grid2d_bias_radius_scan_metrics.csv`, `grid2d_double_peak_candidates.csv`, and `grid2d_bias_radius_heatmap_theta0.pdf`.
- 2D targeted examples: `final_multitimescale_fpt_encounter/artifacts/data/grid2d_typical_double_peak_cases.csv` and `grid2d_typical_double_peak_cases.pdf`.
- 2D local-bias basin pilot: `final_multitimescale_fpt_encounter/code/scan_grid2d_local_bias_basin.py`, `final_multitimescale_fpt_encounter/artifacts/data/grid2d_local_bias_basin_scan.csv`, `grid2d_local_bias_basin_heatmap.pdf`, and `grid2d_local_bias_basin_local_positive_best.pdf`.